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MECHANICS

BY

H. E. SMITH

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MECHANICS

MECHANICS

AN ELEMENTARY STUDY PREPARED FOR THE
USE OF THE MIDSHIPMEN AT THE
UNITED STATES NAVAL ACADEMY

BY

de
H. E. SMITH

Professor of Mathematics, U. S. Navy

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ANNAPOLIS, MD.

THE UNITED STATES NAVAL INSTITUTE

1913

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The Lord Baltimore Press
BALTIMORE, MD., U. S. A.

PREFACE.

This book has been prepared for the use of midshipmen at the U. S. Naval Academy with the view of their requirements of the subject of Mechanics before taking up purely professional subjects. The aim has been to provide a course in Mechanics which will include the principles required in pursuing the professional subjects of Steam and Electrical Engineering, Ordnance, Compass Deviation and Naval Architecture, as these subjects are taught at the Naval Academy.

The text is very brief; first, on account of the short time allowed in which to cover the subject; second, the important principles are not smothered, in a brief work of this kind, by the many details which necessarily accompany a thorough work; and third, the Departments at the Naval Academy, which handle subjects for which a knowledge of Mechanics is necessary, bring out strongly the particular principles involved in their subjects and cover those principles in greater detail than would be warranted in a course of this kind.

Engineers' units have been used throughout the book, principally because the student, who is already somewhat familiar with these units, will have to use them while he remains in the Naval Service. Brief definitions of the other units may be found in the Appendix. It has been the author's observation that many of the students who had excellent marks under the system of absolute units, formerly taught here, frequently even failed to apply properly the principles of Mechanics when confronted with problems stated in engineers' units, the only units one hears mentioned outside of the recitation room. Obviously the absolute unit of force, the poundal, has its uses; but it is used by a comparatively small number of the English speaking people and

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its accuracy does not justify us in practically discarding the ordinary unit of force, the pound. Besides, with a correct understanding of the premises, the pound is just as accurate a measure of force as the poundal. The use of both systems of units in a book of this kind would involve additional explanations for which we have not the space, and also the student is saved the trouble and confusion of learning two systems, for one of which he will find but little use.

In writing this book the subject has been treated with the single object in view of satisfying the needs of the Naval Academy course. For example, the Naval Academy course in Physics covers a number of things usually discussed in books on Mechanics; notably, the pulley, the lever, harmonic vibrations, etc.; also, in the Calculus, the study of which precedes Mechanics at the Naval Academy, centers of gravity and moments of inertia have been discussed. Such things as the midshipmen have already studied are either omitted entirely or very briefly treated. There are brief discussions of centers of gravity and moments of inertia in the Appendix and the general method for harmonic vibrations appears in the text.

Calculus has been used whenever it was found convenient. The problems have been taken from old examinations, with few exceptions. The author's thanks are due to Lieut. W. W. Smyth, U. S. N., for checking the answers to problems, and to him and Prof. H. L. Rice, U. S. N., for criticism.

H. E. S.

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STATICS.

CHAPTER I. *St. 1-9*

DEFINITIONS. VECTOR QUANTITIES. VECTORS.

1. Matter. Mass.—Matter is that of which anything is composed, so that we may take it that everything that occupies space is matter. Matter exists in the solid, liquid and gaseous *states* and in an indefinite number of *forms* in each state. We recognize matter in its different forms by means of our senses; sight, feeling, smell, etc., and the different forms or kinds of matter are called *substances*, as stone, water, hydrogen.

Portions of matter which have definite dimensions are called *material bodies* and, in Mechanics, we compare material bodies by stating the quantity of matter each contains. This quantity of matter which any material body possesses is called its *mass*.

We have then:

Matter is anything that occupies space.

A *material body* is a portion of matter having definite dimensions.

The *mass* of a material body is the quantity of matter it possesses.

2. Force.—We all have a more or less comprehensive idea of what force is. We know it is a push or a pull, a pressure or a suction, a repulsion or an attraction. We know that when a force acts on a material body, motion sometimes results and sometimes does not. We know that when motion *does* result the motion takes place *in the direction* in which the force acts. We know, when motion *does not* result, that there is a resistance to motion which must be as great as the force and *acting in an opposite direction*, and that, were it not for the resistance, motion would

occur. There must be then a *tendency to motion* when a force acts, even though, because of the resistance, no motion takes place. Also, the resistance must be force as it overcomes the effect, the tendency to motion, of which the original force is the cause. We may therefore, tentatively, use the definition.

Force.—That which causes, or resists, motion. *How measured?*

3. Mechanics.—That science or branch of applied mathematics which treats of the conditions of rest, or motion, of material bodies when under the action of force is called Mechanics. *particles or*

Statics.—That part of Mechanics which treats of material bodies *at rest* under the action of force is called statics. *Statics*

Kinetics.—That part of Mechanics which treats of material bodies *in motion* under the action of force is called kinetics. *Kinematics*

Kinematics.—When motion is considered apart from the thing moved and without regard to the force causing the motion, we employ a branch of science called kinematics. *Geometry of motion. "plus time."*

4. Scalar and Vector Quantities.—We may compare two material bodies by stating the amount of matter (the mass) each contains. We may compare two *forces* by stating their magnitudes *and the directions* in which they act. It will be noted that the two masses are to each other as the numbers which indicate their magnitudes, while in comparing the two forces it is necessary to know, in addition to the numbers indicating their magnitudes, the *directions* in which the forces act. Therefore, such quantities as *mass* possess magnitude only, while quantities like *force* possess both magnitude and direction. The former are called *scalar quantities* and the latter *vector quantities*; thus, time, weight, mass, etc., are scalar quantities, and velocity, force, momentum, etc., are vector quantities. *acceleration or impulse, displacement*

We may represent a scalar quantity by drawing a line of as many units in length as the number which represents the magnitude of the scalar quantity, but to represent a vector quantity

we must draw a line representing its magnitude *and draw it in the direction* in which the vector quantity acts. The line representing a vector quantity is called a *vector*; whence, a *vector* is a straight line of as many units length as the magnitude of, and drawn in the same direction as acts, the vector quantity which it represents.

5. Addition and Subtraction of Vectors.—A vector represents by its length the magnitude (to any convenient scale) and shows, by the angle it makes with some fixed line of reference, the direction in which the vector quantity, which it represents, acts.

Two vector quantities are equal when their vectors are of equal length (to the same scale) and parallel.

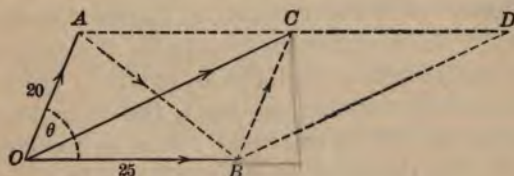


FIG. 1.

If two vector quantities act in the *same* direction, their sum is a vector quantity of magnitude equal to the sum of the magnitudes of the two quantities and acts in the *same direction* as the two quantities. It is represented by a vector parallel to the vectors of the original quantities and of length equal to the sum of their lengths. If the two original vector quantities act in *opposite directions*, their sum is equal to the difference of the magnitudes of the two original quantities and acts in the direction of the greater of the original quantities. The sum of two vector quantities which act at an angle with each other is that vector quantity which would exactly replace both of the original quantities. Thus, if by a vector, OA , we represent the displacement of a point from O to A , Fig. 1, and by another vector, OB , the displace-

ment of a point from O to B , the sum of the two displacements will be OC ; for, if a point starting at O suffers the displacement, OB , and then the displacement, $OA = BC$ (by construction), the point will be at C after the second displacement. Had the point first suffered the displacement, OA , and then the displacement, $OB = AC$, it would again be found at C after the second displacement; therefore, the displacement, OC , will produce the same effect as the two displacements, OA and OB , and this without regard to the order in which OA and OB are taken.

Again, if the two displacements, OA and OB , should act on a point *simultaneously*, the point would actually move along the line OC ; therefore, the sum of the two vectors, OA and OB , is OC , a diagonal of the parallelogram $OACB$.

The *difference* of two vectors may be derived directly from the above; for, CB , is the difference between the two vectors, OC and OB , and it will be noticed that, CB , is, relatively (with reference to the diagonal representing the *sum*), the *other* diagonal of a parallelogram ($OCDB$ in this case). That is, the *given* vectors being drawn from the point O , their *sum* will be represented by the diagonal *between them*, and their *difference* by the diagonal *connecting their ends*, of the parallelogram of which they form two adjacent sides.*

From Trigonometry we have the length of OC from the formula,

$$\overline{OC}^2 = \overline{OA}^2 + \overline{OB}^2 + 2 \cdot OA \cdot OB \cdot \cos \theta. \quad (1)$$

The other diagonal is given by

$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 - 2 \cdot OA \cdot OB \cdot \cos \theta. \quad (2)$$

* It is worth noting that the difference of two vectors is the sum of one of them and the other *reversed* in direction; for, referring to Fig. 1, the sum of OC and a vector represented in magnitude by OB , but of reversed direction, is equal to BC . Assume the point B as origin, the vector $BD = OC$, in magnitude and direction. Reverse the direction of OB , add and the sum is BC . Thus $OC + (-OB) = BC$.

The angle COB , between OC and OB , is given by

$$\tan COB = \frac{OA \sin \theta}{OB + OA \cos \theta} \cdot \text{prove this} \quad (3)$$

If by arrowheads we indicate *the directions* of the displacements OA and OB (Fig. 1), it will be observed that the *sum* of the two displacements is represented by the vector, OC , and *acts* in the direction *away* from the point, O (the origin for the two vectors), along the diagonal between OA and OB ; while, referring to the parallelogram, $OCDB$, it will be noted that *the difference* between the vectors, OC and OB , is the vector, BC , acting in the direction from B to C ; that is, directed toward the further end of the *greater* displacement, OC . The direction of the sum or difference of two vectors may always be indicated in this way, *if the vectors be drawn so that the direction of the vector quantities is indicated by an arrowhead pointing away from the point selected as origin for the vectors*: for example, the difference between the two vectors, OA and OB , is the vector, AB , whose vector quantity acts in the direction from A to B as indicated.

The *sum* of two vector quantities is called the *resultant vector quantity*, or, simply *the resultant*. The quantities themselves are called the *component vector quantities*, or, simply *the components*.

6. Composition and Resolution of Vector Quantities.—From the preceding article, it is evident that, if two vector quantities, when added together, have the same effect as the single vector quantity which represents their sum, we can separate any single vector quantity into two vector quantities, which shall act in any two given directions, and whose sum shall equal the single vector quantity. The operation of finding the single vector quantity which will replace two vector quantities is called the *composition* of vector quantities; that of separating a single vector quantity into two vector quantities whose sum shall equal the single vector quantity is called the *resolution* of vector quantities. The above

operations are limited, (1), to combining two quantities into one which shall be equal to the two, or, (2), to separating one quantity into two which shall be equal to the one; but, it is obvious that, in the first case, either, or both, of the two original quantities may be the sum of two other quantities; and in the second case, that either, or both, of the parts, of which the original quantity is the sum, may be further separated into two still different parts. Therefore, any number of vector quantities may be combined so

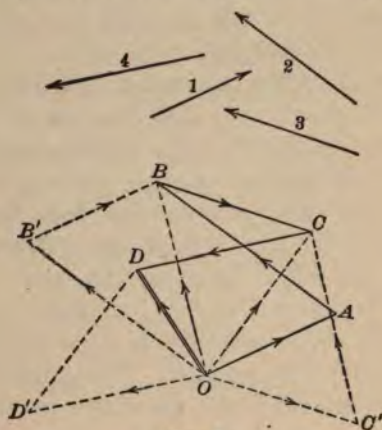


FIG. 2.

that their sum is a single vector quantity and may be represented by a single vector; or, a single vector quantity may be separated into any number of vector quantities, acting in any given directions, and whose sum shall equal the single original vector quantity. The operation as indicated above (taking the quantities two by two) is tedious and may be considerably shortened as follows: Given the vector quantities which are represented by the vectors, 1, 2, 3 and 4, Fig. 2. Lay off from any point, O , in its direction the vector OA equal to 1. From the end of this vector lay off AB equal to 2. From the end B of AB lay off BC equal

to 3. From C lay off CD , equal and parallel to 4. The line OD , connecting the end of this last vector with O , will be the sum of the vectors and is a vector which represents the sum both in magnitude and direction. For, the sum of OA and AB , from the parallelogram $OABB_1$, is OB and the sum of OB and BC , from the parallelogram $OBCC_1$, is OC . The sum of OC and CD from the parallelogram $OCDD_1$, is OD . Therefore, OD , which connects the end of the last vector with O , is the sum of the given vectors; or OD is the resultant vector of the component vectors 1, 2, 3 and 4.

The direction in which this resultant acts is from O toward D , which is readily seen by following the method of the preceding article.

By reversing the above method we may resolve any vector quantity into $n+1$ vector quantities, n of which may be made equal to n given components; for, let OD be the vector representing the vector quantity to be divided and let 1, 2, 3 and 4, be vectors representing vector quantities of given magnitude and direction, but whose sum is not necessarily *equal* to the vector quantity represented by OD ; then, from O lay off OD_1 , equal to the given vector 4; the difference between OD and OD_1 , or DD_1 , is the remainder after subtracting one of the given vectors. From O draw the line OC , which completes the parallelogram OD_1DC , and subtract another of the given vectors from OC . This leaves us another remainder, and so on. Having subtracted the last of the given vectors the last remainder together with the given vectors will have a sum equal to the vector which was to be divided.

If, instead of being given n vectors (which indicate magnitude as well as direction) we are given n *directions* only in which the component quantities are to act, we may divide the original quantity into n parts whose sum will equal the original quantity; for, assuming *direction only* to be indicated by 1, 2, 3 and 4,

Fig. 3, and desiring to divide the quantity represented by the vector, OE , into four quantities which shall act in the directions indicated by 1, 2, 3 and 4, Fig. 3, we draw from O , in the direction of 1, the line OA . From any point on OA , say A , we draw AE , which represents the remainder left after subtracting the quantity represented by OA ; OA being the vector representing the first of the required quantities. From A draw a line in the direction indicated by 2; the line from any point on it, say B to E , will

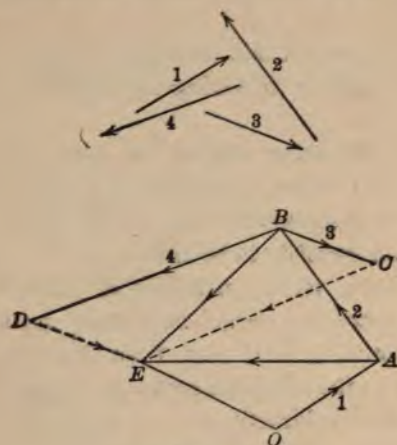


FIG. 3.

represent the remainder left after subtracting from AE the quantity represented by AB . The vector AB represents the second of the required quantities. There being now left only two directions for which we require vectors, we draw from both B and E lines in the directions indicated by 3 and 4. The lines from E will intersect the lines from B in two points, C and D , such that BC and BD will be vectors representing the third and fourth of the required quantities, respectively; for, the four vectors, OA , AB , BD and DE , have for their sum OE , as desired.

*or Cartesian, as against vectorial method
coordinates in general*

7. Rectangular Resolutions.—The most convenient way of dividing a vector quantity into two vector quantities whose sum shall equal the original quantity, is to resolve the original quantity into two quantities whose directions shall be at an angle of 90° with each other. This operation is called *rectangular resolution*, and, as the two directions are already known and the magnitudes obtained from a simple trigonometric formula, its simplicity commends its use.

Referring to Fig. 4, let OA represent a vector quantity whose direction makes the angle θ with the line OX . Then the sum of

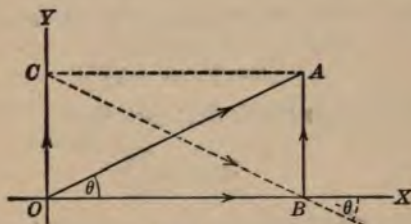


FIG. 4.

the two vector quantities represented by the vectors OB and BA will equal OA . The magnitudes of these vectors are

$$BA = OA \sin \theta, \quad (1)$$

$$OB = OA \cos \theta, \quad (2)$$

and the directions are already known to be along and perpendicular to OX .

Taking the reverse, having given the two vectors, OB and BA , acting at an angle of 90° with each other, their sum is equal in magnitude to the square root of the sum of their squares and makes an angle with OX equal to θ , where

$$\theta = \tan^{-1} \frac{BA}{OB} = \cos^{-1} \frac{OB}{OA} = \sin^{-1} \frac{AB}{OA}, \text{ etc.} \quad (3)$$

The magnitude of their sum is

$$OA = OB \sec \theta = BA \operatorname{cosec} \theta = \sqrt{OB^2 + BA^2}. \quad (4)$$

Also, from the figure, the magnitude of their difference is equal to the magnitude of their sum, but acts along a line making an angle of $-\theta$ with OX and in the direction toward the end of the greater of the two vectors.

Having a number of vector quantities, their resultant is easily found by resolving them all *along* and *perpendicular* to any fixed line, say OX . Indicating by Σx the sum (algebraic) of the resolved parts along OX and by Σy the sum of the resolved parts perpendicular to OX , the magnitude of the resultant is

$$R = \sqrt{\Sigma x^2 + \Sigma y^2}, \quad (5)$$

and it makes an angle θ with OX , such that

$$\tan \theta = \frac{\Sigma y}{\Sigma x}. \quad (6)$$

With rectangular resolution the components are called *resolved parts*. Where data is given with reference to rectangular co-ordinates the resolutions are usually made along the co-ordinate axes, and we then speak of the "resolved part along the X (or Y) axis."

8. Displacements. Vectors in Space.—Displacements have been used in the preceding examples of vector addition and subtraction because a displacement is the simplest of vector quantities and needs but little explanation. If a point starts at A and arrives at B , the displacement of the point is the *straight line* connecting A with B and is independent of the path traveled. For example; a point moves from A , along the arc of a circle, to B . The displacement is the *chord* AB . The *direction* of the displacement is from A toward B , and the *magnitude* is the distance from A to B , measured along the chord.

Vectors, representing vector quantities, may be laid down anywhere. They may be moved from place to place if care is always taken to keep their directions and magnitudes unchanged, *i. e.*, the vector in the new position must be of equal length and parallel to the vector in the old position. Thus a number of vectors may always be laid down from a common origin. Any convenient scale may be used for indicating the magnitudes, but the *same scale* must be used for *all vectors* that are considered at any one time.

In the preceding the vector quantities have always been taken as lying all in one plane, but the results obtained are readily extended to vectors in space; for, we may pass a plane through any two vectors, then, a plane through the resultant of these two and another vector, etc.; or, by resolving a vector along *three* co-ordinate axes instead of two, and combining the three resolved parts to make up the resultant. The method is that of Analytic Geometry. The resultant will be given by the formula

$$R^2 = \overline{\Sigma x}^2 + \overline{\Sigma y}^2 + \overline{\Sigma z}^2 \quad (1)$$

and calling the angles the resultant makes with the X , Y and Z axes, α , β , γ , respectively,

$$\cos \alpha = \frac{\Sigma x}{R}, \quad \cos \beta = \frac{\Sigma y}{R}, \quad \text{and} \quad \cos \gamma = \frac{\Sigma z}{R}. \quad (2)$$

9. Problem.—Illustrating the convenience of the method indicated in the latter part of Art. 7; suppose it is required to find the resultant displacement when a point is acted upon by the following seven displacements, all of which are in the same plane: 5, $2\sqrt{3}$, 6, 8, $5\sqrt{2}$, 16 and $3\sqrt{2}$ feet, making the following angles respectively with a fixed line in the plane: 0° , 30° , 60° , 120° , 225° , 300° , 315° . The first thing to do is to plot the displacements as shown in Fig. 5. Then resolve each displacement along and perpendicular to the fixed line in the plane, and call this line the X axis. (If the angle the resultant makes with a particular displacement or line is desired, plot the vectors so that *that par-*

*these were
small
vectors*

ticular one lies in the X axis.) Put the resolved parts, with their correct signs prefixed (observe that each component gets its

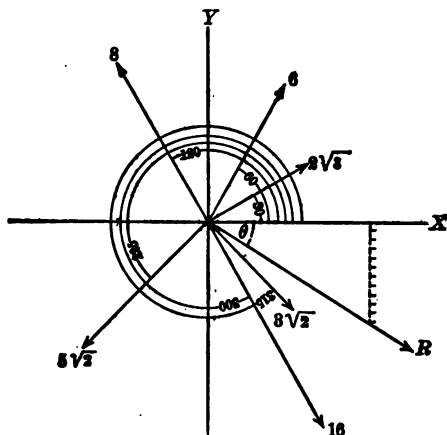


FIG. 5.

X components.			Y components.		
1		5			0
2	$2\sqrt{3} \cos 30^\circ =$	3	$2\sqrt{3} \sin 30^\circ =$	$\sqrt{3}$	
3	$6 \cos 60^\circ =$	3	$6 \sin 60^\circ =$	$3\sqrt{3}$	
4	$8 \cos 120^\circ =$	-4	$8 \sin 120^\circ =$	$4\sqrt{3}$	
5	$5\sqrt{2} \cos 225^\circ =$	-5	$5\sqrt{2} \sin 225^\circ =$	-5	
6	$16 \cos 300^\circ =$	8	$16 \sin 300^\circ =$	$-8\sqrt{3}$	
7	$3\sqrt{2} \cos 315^\circ =$	3	$3\sqrt{2} \sin 315^\circ =$	-3	
$\Sigma x = 13$			$\Sigma y = -8$		

$$R = \sqrt{\Sigma x^2 + \Sigma y^2} = \sqrt{233}. \quad \tan \theta = \frac{\Sigma y}{\Sigma x} = \frac{-8}{13} \therefore \theta = \tan^{-1} \frac{-8}{13}.$$

correct sign from the function of the angle the displacement makes with the X axis), in two columns, as shown, and add algebraically. The sums, which we call Σx and Σy respectively, are the X and Y

components of the resultant displacement. The signs of Σx and Σy clearly indicate the quadrant in which the resultant lies, and the tangent of the angle the resultant makes with the X axis is the quotient of Σy divided by Σx .

This method may be regarded as *the general method* and is to be used for all kinds of vector quantities.

Problems I.

1. A point suffers a displacement of 25 ft. and then one of 20 ft. at right angles with the first. What is the resultant displacement and the angle it makes with the first displacement?

2. If the displacements of (1) are at an angle of 60° with each other, what is the resultant displacement and its angle with the first?

3. A point suffers three displacements of 9, 10 and 11 ft. respectively, each making an angle of 120° with each of the others. What is the resultant displacement and the angle it makes with the 9 ft. displacement?

4. A ship is carried in one hour, by the current 3 mi. N. E., by her screw, 10 mi. N. What is the magnitude and direction of the resultant displacement? *i. e.*, the course and distance made good?

5. Two vessels leave port at the same time. One steams N. E. at 15 kts., the other N. at 10 kts. The current is 3 kts. N. W. How far apart are they at the end of an hour, and how do they bear from each other?

6. After suffering two displacements at right angles with each other a point is found at B , 100 ft. from A , where it started. What were the two displacements if the point was never more than 48 ft. from AB ?

7. If the two displacements of (6) were in the ratio of 3 to 5, and made an angle of 60° with each other, what were their magnitudes?

8. A sailing vessel which can sail within 4 pts. of the wind, is

found at the end of one hour, 10 mi. to windward after tacking 3 times. If the wind was N. and she made twice the distance on the 2d and 3d tacks as she did on the 1st and 4th, what were the courses and distances sailed?

9. A point suffers 5 displacements as follows: 10 ft. E., 12 ft. N. E., 3 ft. N., 7 ft. N. 60° W. and 14 ft. S. W. Find course and distance made good.

10. After 5 displacements a point is 100 ft. N. W. of where it started. Four of the displacements were: E. 16 ft., N. 40 ft., S. W. 72 ft. and S. E. 46 ft. What was the magnitude and direction of the 5th displacement?

CHAPTER II.

FORCES. COMPOSITION AND RESOLUTION. MOMENTS OF FORCES.

10. Force.—If a force acts on a material body and the body is moved, the motion takes place in the direction in which the force acts. A straight line passing through the point at which the force is applied, in the direction in which the motion starts (the direction of the force) is called the “line of action” of the force. The point at which the force is applied is the “point of application” of the force. A force may act at any point in its line of action; as, for example, a weightless cord of *any length* may be used to support a mass; in which case, the force is the attraction of the earth for the mass (the weight), the line of action of the force is coincident with the cord and the point of application is the point at which the cord is held (the end opposite to the one on which the mass is hanging), for this is the point where the downward force of the earth’s attraction is resisted; or rather, it is the point at which the mass is supported; therefore,

The *line of action* of a force is a straight line of indefinite length, which passes through the point of application of the force in the direction in which the force acts.

A force is fully described if we have given its magnitude and the direction of its line of action. Therefore, possessing magnitude and direction, force is a vector quantity.

On account of the force of gravity (which always acts) it is impossible to relieve any material body from the action of all force. If, however, we could suspend a body, in a vacuum, by an inextensible cord of infinite length, that body would be free of all *horizontal* forces and could move freely in a horizontal plane; for, the action of gravity and the supporting force along the cord

would be normal to the horizontal plane and would therefore have no effect on any horizontal motion of the body. Should a body so suspended be acted upon by a horizontal force, however small, it would move along the line of action of that force. To keep it from moving a force of equal magnitude acting in the opposite direction to the first force would be necessary. No matter how many forces act on the body, each one must be balanced in this way if the body is to remain at rest. Such conditions (all forces balanced), extended to all forces acting in whatever plane, are what we have to consider in statics as we have defined the word. A body in the above condition is said to be in *equilibrium*; that is, it will remain at rest indefinitely, but it is always ready to move in response to any new force which may be applied. *pressure*

right to
apply bal-
anced forces

11. A Unit of Force and One of Mass.—It was stated in Art. 1 that the mass of anything was the quantity of matter it possessed. In order to compare masses it is necessary to have some standard unit mass by which the relative magnitudes may be measured. Most countries have such a standard unit mass and it is usually a piece of platinum, carefully preserved at the seat of government. The unit of mass is thus a selected unit. The units of force and mass are connected, as we shall find in studying kinetics, and therefore, the unit of force was taken as *that force with which the mass of the earth attracts this selected standard mass*. This force, in England and the United States, is known as *the pound*. — ~~The standard mass is that mass which the earth attracts with the force of one pound.*~~ With these units, to compare the masses of two bodies we simply weigh the bodies, the ratio of the weights (the forces) being equal to the *ratio* of the masses. To compare two forces, we compare their magnitudes *in pounds* and the *directions* of their lines of action.

The subject of units will be considered more fully under kinetics.

12. Combinations of Two or More Forces.—Names have been given to certain combinations of simultaneously acting forces.

*The ~~English~~ *English* unit of mass possesses g (32.2) times the quantity of matter in this standard mass.

the unit of mass contains 32.2 times as much
as 1 lb mass. Therefore the unit is 32.2 lb

Forces whose lines of action all lie in the same plane are called *co-planar forces*, and those whose lines of action do not lie in the same plane are called *non-coplanar forces*.

A number of forces which act at the same point and whose lines of action are coincident, when acting in the *same* direction, are called *conspiring forces*; and whether acting in the same or in opposite directions, they are called *collinear forces*.

A number of forces which act at a point, that is whose lines of action intersect at the point of application, are called *concurring forces*.

Parallel forces are those whose lines of action are parallel.

Two equal and opposite forces, *not collinear*, whose lines of action are parallel form a *couple*. The *arm of a couple* is the perpendicular distance between the lines of action of the forces.

13. Vector Representation.—Force, being a vector quantity, may be graphically represented by laying off on a line, parallel to the line of action of the force, as many equal divisions as indicate the magnitude of the force. By thus representing force *all the methods of Chapter I*, for vector addition, subtraction, etc., will be equally serviceable when applied to forces represented by vectors.

For example: two forces of 20 and 25 pounds, being represented respectively, in magnitude and direction, by *OA* and *OB* of Fig. 1, could be replaced by a single force, represented both in magnitude and direction by *OC*. Moreover, the single force *OC* would have, in all respects, the same effect as the two forces represented by *OA* and *OB*. The calculations for magnitude and direction are those given in Art. 5; the magnitude of the resultant force being given by

$$\overline{OC}^2 = \overline{20}^2 + \overline{25}^2 + 2 \times 20 \times 25 \cos \theta,$$

and the direction, with reference to that of *OB*, being given by

$$\tan COB = \frac{20 \sin \theta}{25 + 20 \cos \theta}.$$

14. Composition and Resolution of Forces in Space.—When concurring forces are non-coplanar the following method may be used to obtain their resultant:

Suppose a particle is acted upon by three forces whose lines of action are each at right angles with the other two. Referring to Fig. 6, let x , y and z represent in magnitude and direction the

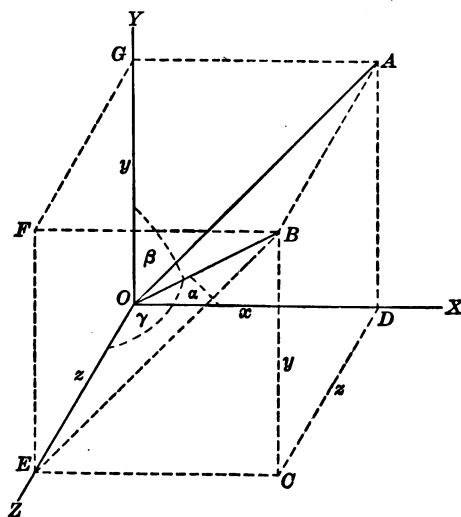


FIG. 6.

three forces. From the figure it is clear that the resultant of x and y will be OA , and also that OB will be the resultant of OA and the remaining force, z . Therefore, OB represents both in magnitude and direction the resultant of the three forces.

Assuming O as origin, and calling the angles the resultant, OB , makes with the co-ordinate axes, α , β and γ , respectively, it is obvious that;

$$R \cdot \cos \alpha = x; R \cdot \cos \beta = y; \text{ and } R \cdot \cos \gamma = z, \quad (1)$$

squaring and adding equations (1) we have

$$x^2 + y^2 + z^2 = R^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma),$$

but it is obvious from the figure that

$$R^2 = \overline{OC}^2 + y^2 = x^2 + y^2 + z^2, \quad (2)$$

therefore;

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \quad (3)$$

α , β and γ are called the *direction angles* and $\cos \alpha$, $\cos \beta$ and $\cos \gamma$, the *direction cosines* of the resultant of the co-ordinate forces x , y and z .

Having a number of concurrent, non-coplanar forces described by their magnitudes and their direction angles (or other angles from which the direction angles may be derived) we may now find their resultant by a method similar to that of Art. 7 for their resultant will be the square root of the sum of the squares of their algebraically added co-ordinate forces, or

$$R = \sqrt{\Sigma x^2 + \Sigma y^2 + \Sigma z^2}, \quad (4)$$

and its direction cosines will be

$$\cos \alpha = \Sigma x / R; \cos \beta = \Sigma y / R; \cos \gamma = \Sigma z / R. \quad (5)$$

15. The Effective Part of a Force in a Given Direction.—

Switch engines are usually provided with a timber strut by means of which they can push cars which are on a track parallel to the one they occupy. When in operation this strut is placed diagonally between the car and locomotive. The force of the locomotive's push is in the direction of the track, but there is a force which acts along the strut, and a force which tends to push off to the opposite side of the track; the track resistance to this latter push causes the grinding between the track and the wheel flanges of the engine. The resultant (the push in the strut), calling the angle the strut makes with the track, θ , is $P \sec \theta = S$.

Again, the part of the push which is acting along the strut tends to push the car in the direction the strut points and, were it not for the wheel flanges of the car, would do so. The effective part of this, that which pushes the car *along* the track, is the strut force S multiplied by the *cosine* of the angle, or, as the angles are equal, the tracks being parallel, the effective force which acts to push the car ahead is $P \cdot \cos \theta \sec \theta = P$.

In the first case the push of the locomotive is the resultant of the force in the strut and the force which causes the grinding between the wheel flanges of the engine and the track; in the second case the *force in the strut* is the resultant of the force which shoves the car along the track and the one which causes *its* wheel flanges to grind against the track. In both cases the effective force is the *strut* force multiplied by the cosine of the angle θ ; or, the *effective part* (resolved part) of a force, S , along a line making an angle θ with S , is $S \cos \theta$. *the other is the lost component*

16. Moment of a Force About a Point.—The moment of a force about a point is the product of the magnitude of the force and the perpendicular distance from the point to the line of action of the force. This moment is a measure of the effort a force would make to rotate a body, fixed at the given point, but free to turn about it; for the effort is proportional to the magnitude of the force and also to the distance of the force from the point (principle of the lever). *measured in lb ft or oz in*

If the moments of several forces are to be considered, a sign, plus or minus, should be prefixed in accordance with the *direction* of the turning effort. Custom has made the effort in a counter-clockwise direction the positive, and the clockwise, the negative, direction of rotation.

17. Varignon's Theorem.—The algebraic sum of the moments of two concurring forces, about any point in their plane, is equal to the moment of their resultant about the same point.

Let P and Q , Fig. 7, be the two forces and R their resultant;

let M be any point, and p , q and r be perpendiculars from the point upon the lines of action of the forces; then, taking OM for axis of Y and OX , perpendicular to OM , for the X axis,

$$P \cdot \cos \alpha + Q \cdot \cos \beta = R \cdot \cos \gamma.$$

Multiplying by OM ,

$$P \cdot OM \cdot \cos \alpha + Q \cdot OM \cdot \cos \beta = R \cdot OM \cdot \cos \gamma.$$

But,

$$OM \cdot \cos \alpha = p; OM \cdot \cos \beta = q; \text{ and } OM \cdot \cos \gamma = r.$$

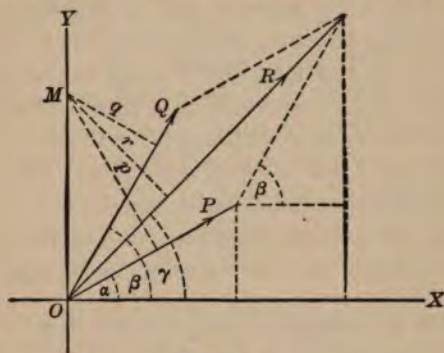


FIG. 7.

Hence

$$Pp + Qq = Rr,$$

which proves the theorem. *Extension to any # of forces.*

18. Moment of a Force About a Line.—The moment of a force about a line is a measure of the effort of the force to rotate a body which is free to turn about the line but not to slide along it.

The line about which the moment is taken is called the *axis of moments*.

If the force is ^{in a plane} perpendicular to the axis of moments, the moment of the force is equal to the product of the magnitude of the force and the shortest distance between the line of action of

the force and the axis of moments. This "shortest distance" must be measured along the line which is perpendicular both to the line of action of the force and the axis of moments. *Rule:*

If the line of action of the force be inclined to the axis of moments, we must resolve the force into two components, one parallel to the axis of moments, and the other perpendicular to it. (The perpendicular component must necessarily be in the plane of the force and its component parallel to the axis of moments.) It is clear that the perpendicular component only will be effective to rotate the body. The measure then of an oblique force to rotate a body about a line (the axis of moments), or, the moment of this force about the axis of moments is equal to the product of that resolved part of the force which is perpendicular to the axis of moments and the shortest distance from the axis of moments to the line of action of that resolved part. This "shortest distance" is the same as the shortest distance between the line of action of the force and the axis of moments. A sign should be prefixed, indicating the direction of the turning effort. *mom. = 0* $\begin{matrix} 1. \perp \\ 2. \parallel \\ 3. \neq 0 \end{matrix}$

19. Moment of a Couple.—The moment of a couple about any point in its plane is equal to the product of *one* of the forces and the perpendicular distance between the lines of action of the two forces forming the couple; for,

Let the couple be as represented by the two *equal* forces, F_1 and F_2 in Fig. 8, the perpendicular distance between them being b ; let M be any point at any distance, a , from one of the forces. The moment of F_1 about the point is $F_1(b+a)$, and that of F_2 is $-F_2 \cdot a$. The algebraic sum of these moments is $+Fb$. Which was to be proved. *How to picture it*

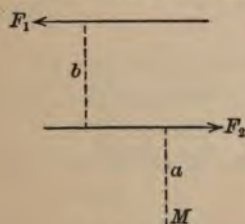


FIG. 8.

A body acted upon by a couple will tend to rotate and no *single* force can counteract the tendency. Another couple whose moment is equal

and *opposite* to the one already acting, would be necessary to keep the body at rest. *crank*.

20. Problems.—(a) Three concurring, co-planar forces of 1, 2 and 3 lbs., respectively, act at angles of 120° with each other. What is their resultant and the angle it makes with the 3 lb. force?

Desiring the angle the resultant makes with the 3 lb. force, we put the vector for that force in the axis of X .

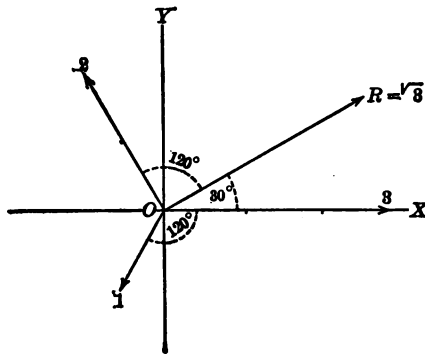


FIG. 9.

X components.	Y components.
3	0
$2 \cdot \cos \cdot 120^\circ = -1$	$2 \cdot \sin \cdot 120^\circ = \sqrt{3}$
$1 \cdot \cos \cdot -120^\circ = -1/2$	$1 \cdot \sin \cdot -120^\circ = -\sqrt{3}/2$
$\Sigma X = 3/2$	$\Sigma Y = \sqrt{3}/2$

$$R^2 = \overline{\Sigma x}^2 + \overline{\Sigma y}^2 = 3. \quad R = \sqrt{3}.$$

$$\tan \theta = \Sigma Y / \Sigma X = 1 / \sqrt{3}.$$

$\tan^{-1} 1/\sqrt{3} = 30^\circ$ which is the angle that the resultant makes with the 3 lb. force.

(b) The angle between the lines of action of two forces, P and Q is ϕ . What is the value of their resultant?

By Art. 5 (1)

$$R^2 = P^2 + Q^2 + 2PQ \cdot \cos \phi.$$

The angle it makes with one of the forces, P , will be given by equation (3) of Art. 5, whence,

$$\tan \theta = \frac{Q \sin \phi}{P + Q \cos \phi}.$$

(c) Three concurrent forces act in lines, each of which is at an

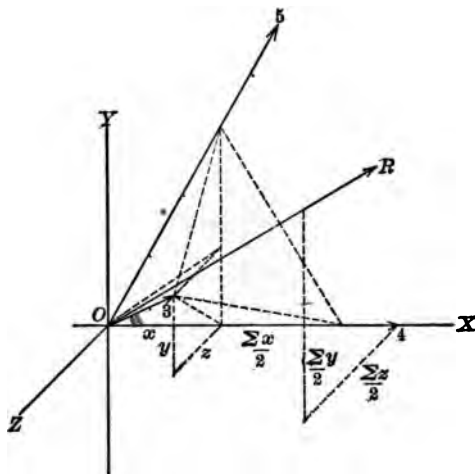


FIG. 10.

angle of 60° with the other two. If the forces are 3, 4 and 5 lbs., respectively, find their resultant and locate its line of action (see Fig. 10).

Drawing the vectors so that two of them, say those for the 4 and 5 lb. forces, will lie in the plane of XY , we have at once *that a*, the angle the 3 lb. force makes with the X axis is 60° . If

we pass a plane so as to cut off three divisions on each vector, we will have outlined an equilateral pyramid, whose vertex, or the extremity of the 3 lb. vector, will lie at a perpendicular distance of $\sqrt{6}$ divisions from the plane of XY and at a perpendicular distance of $\sqrt{3}/2$ divisions from the plane of XZ ; or, the z component of the 3 lb. force is $\sqrt{6}$ and the y component is $\sqrt{3}/2$ pounds. The x component is $3 \cdot \cos 60^\circ = 3/2$ pounds.

Taking the algebraic sum of the components of all three forces:

Force.	X components.	Y components.	Z components.
4	4	0	0
5	$5 \cos 60^\circ = 5/2$	$5 \sin 60^\circ = 5\sqrt{3}/2$	0
3	$3/2$	$\sqrt{3}/2$	$\sqrt{6}$
	$\Sigma X = 8$	$\Sigma Y = 3\sqrt{3}$	$\Sigma Z = \sqrt{6}$

In accordance with Art. 14,

$$R^2 = \Sigma X^2 + \Sigma Y^2 + \Sigma Z^2 \text{ or } R = \sqrt{64 + 27 + 6} = 9.85 \text{ (nearly)}$$

and

$$\cos \alpha = 8/9.85 \quad \cos \beta = 3\sqrt{3}/9.85 \quad \cos \gamma = \sqrt{6}/9.85$$

which is the magnitude of the resultant together with its direction cosines.

(d) The wind is from the North and blowing with a force P . An ice boat heads N. 60° E., and the sail, considered flat and in the vertical plane, makes an angle of 30° with the boat's runners. What is the effective force driving the boat ahead? *

Ignoring the air friction from the component parallel to the sails; the normal pressure on the sail is $P \cdot \cos^2 60^\circ = \frac{P}{4}$. This force acting normal to the sails has a component parallel to the

* Note that only half of sail area intercepts wind.

runners which is equal to $\frac{P}{4} \cdot \cos 60^\circ = \frac{P}{8}$. This is the force which is effective in driving the boat ahead.

(e) The following co-planar forces act about a point: 3 lbs. acts toward the right at a distance of 2 ft. above the point; 5 lbs. acts downward, 3 ft. to the left of the point; 2 lbs. acts upward, 8 ft. to the left; 1 lb. acts upward, 10 ft. to the right; and 4 lbs. acts toward the right, 3 ft. below the point. Find the effort of these forces to rotate about the point and the direction in which it acts.

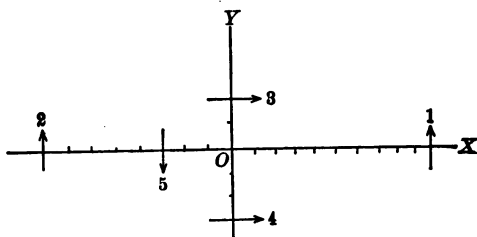


FIG. 11.

Taking moments about the point,

$$-3 \times 2 + 5 \times 3 - 2 \times 8 + 10 \times 1 + 4 \times 3 = 15 \text{ lbs.-ft.}$$

This result is called 15 *pounds-feet*. Care must be taken to distinguish between it and the unit of work, which we will find later on is called "foot-pounds." The result above means that the effort to turn about the point is the same as that a one pound force, at a distance of 15 feet, would exert; or, as that a 15 pound force, at a distance of one foot, would exert.

The *direction* of the effort is counter-clockwise, or positive.

Problems II.

1. Concurrent, co-planar forces of 2, 4, 6 and 8 lbs. act in lines making angles of 30° , 120° , 225° and 270° , respectively, with a

fixed straight line. Find their resultant and the angle it makes with the fixed straight line.

2. Concurrent, co-planar forces of 10, 8, 2 and 12 lbs. act at angles of 45° with each other. Find their resultant and the angle it makes with the 10 lb. force.

3. Forces of 5, 6 and 7 lbs. making equal angles with one another, lie in the same plane and act upon a particle. What is the least force that will keep the particle at rest and what angle will it make with the 6 lb. force?

4. Three ropes are attached to a bolt in the ceiling. The bolt being 12 ft. vertically above the plane of a man's pull. The ropes are 15, 20 and 20 ft. long and lead off respectively, in the directions, North, Southeast and West from a point directly below the bolt. A man pulls each rope with a force of 100 lbs. Find the magnitude and direction of the pull on the bolt in the ceiling.

5. Forces of 3 and 4 lbs. act at an angle of 60° with each other. Find the resultant and the angle it makes with the 4 lb. force.

6. Forces of 5 and 9 lbs. act at an angle of 120° with each other. Find the resultant and the angle it makes with the 9 lb. force.

7. Referring to Fig. 6, forces act in the plane of XZ along the following lines: from O to C a force of 9 lbs.; from C to E , 6 lbs.; from E to D , 5 lbs.; and from C to D , 4 lbs. If the dimensions are $OD=3$ ft., $OE=4$ ft., find the moment of the forces about the point O .

8. Referring to Fig. 6, forces act along the lines of the parallelogram as follows: from B to A , 3 lbs.; B to F , 5 lbs.; E to D , 7 lbs.; A to E , 9 lbs.; and from A to G , 8 lbs. If the dimensions are $OD=4$ ft., $OG=5$ ft., and $OE=3$ ft., find the moment of the forces about the line OY .

9. The center pole of a tent is supported by three guys each 50 ft. long, and leading off from a point 40 ft. above the horizontal plane, in directions, North, N. 120° E., and N. 150° W.

If the tension of each guy is 100 lbs. find the resultant pull on the pole and its direction.

10. An "L" lever has arms which are 3 ft. and 1 ft. in length and inclined to each other at an angle of 120° . If the plane of the lever is vertical and 50 lbs. acts in its plane and at an angle of 30° with the long arm and at its end, what force acting perpendicular to the short arm will be necessary to hold the lever at rest?

CHAPTER III.

PARALLEL FORCES. CENTER OF PARALLEL FORCES.

21. The Resultant of Two Parallel Forces.—Let P and Q of Fig. 12 represent two parallel forces. If along any line, AB , which intersects the lines of action of P and Q , we introduce two equal and opposite forces, F , we will not disturb the action of P and Q , for the forces, F , counteract each other. Referring to the figure, the resultant of P and the left force, F , will be AD ; and

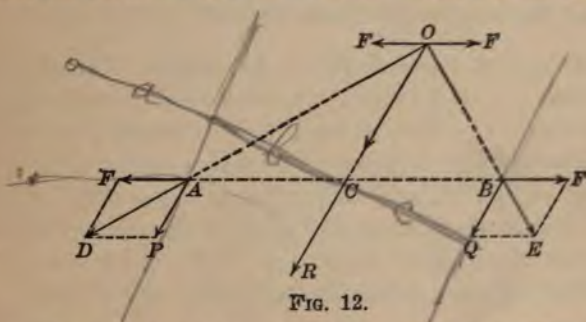


FIG. 12.

the resultant of Q and the right force, F , will be BE . The intersection of the lines of action of these resultants is O , and if at this point we resolve them back into their components, we will have the forces, F , neutralizing each other and the forces P and Q acting together along a line which is parallel to their original lines of action. Hence the magnitude of the resultant is equal to the sum of the forces, and its line of action is parallel to the lines of action of the original forces.

Referring to the figure, from the similar triangles, DAP and AOC ,

$$P:F::OC:CA,$$

and from the similar triangles, QBE and COB ,

$$Q : F :: OC : CB,$$

by division

$$P : Q :: CB : AC, \quad (1)$$

or the resultant of two parallel forces is equal to their algebraic sum and its line of action is parallel to the line of action of the forces and divides any line between the forces into segments which are inversely proportional to the forces.

In case the forces act in opposite directions, the line which intersects their lines of action must be produced, and it is then divided *externally* into segments which are inversely proportional to the forces (prove by above method).

22. Resultant of Parallel Forces by Taking Moments.—A much simpler way of getting the resultant and the position of its line of action, now that we know that its magnitude is equal to the algebraic sum of the forces, is by taking moments.

Referring to Fig. 12, and taking moments about the point A (moments may be taken about *any* point in the plane of the two forces without changing the result, but the point A is convenient as it brings directly the proportion in the preceding article),

$$R \times AC = Q \times AB = Q \times (AC + CB), \text{ angles}$$

but $R = P + Q$, hence

$$(P + Q) \times AC = Q \times (AC + CB),$$

or,

$$P : Q :: CB : AC$$

which is the same result as before. In fact this is the general method; for, the resultant of a number of parallel forces is equal to the algebraic sum of the forces, and if the forces are all in one plane, the sum of the moments of all the forces about any point

in the plane is equal to the moment of their resultant about that point. In other words,

$$R = \Sigma P,$$

then if $\bar{r}$ is the distance from the point chosen as origin to the line of action of R ,

$$\bar{r} = \frac{\Sigma Pr}{\Sigma P},$$

where ΣPr is the algebraic sum of the moments of the forces about the origin. A special case is that when $R = \Sigma P = 0$ and ΣPr is finite. Such conditions indicate a couple whose moment is equal to ΣPr . (See Art. 23.)

23. Parallel, Non-Coplanar Forces.—When the forces are not

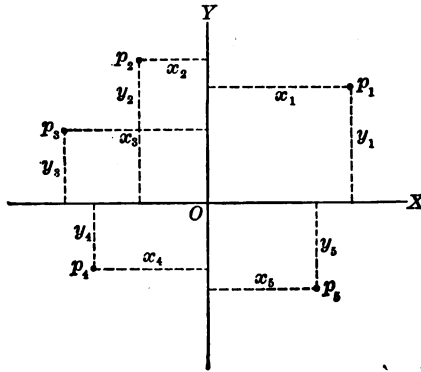


FIG. 13.

*and in that plane as
set of axes*

all in one plane, imagine a plane passed perpendicular to the lines of action of all the forces. Referring to Fig. 13, let the plane of XY be this plane; let p_1, p_2, p_3, p_4 and p_5 be the forces, all acting in the same (positive) direction, and let $(x_1, y_1), (x_2, y_2)$, etc., be the co-ordinates of the points at which the lines of action of the

forces pierce the XY plane. Then, taking moments about the axis of Y (moments about a *line*) we have

$$p_1x_1 - p_2x_2 - p_3x_3 - p_4x_4 + p_5x_5 = \Sigma Px,$$

and moments about the axis of X , give

$$p_1y_1 + p_2y_2 + p_3y_3 - p_4y_4 - p_5y_5 = \Sigma Py.$$

The algebraic sum of the forces is equal to ΣP , and

$$\bar{x} = \frac{\Sigma Px}{\Sigma P}, \quad \text{and} \quad \bar{y} = \frac{\Sigma Py}{\Sigma P}, \quad (1)$$

which are the co-ordinates of the point where the line of action of the resultant pierces the plane. The distance, $\bar{r}$, from the origin to point where the resultant, ΣP , pierces the plane is given by

$$\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2}.$$

Care must be taken to *give to the moments their correct sign* as explained in Art. 16.

A special case is presented when the algebraic sum of a number of parallel forces not in the same plane is zero and the algebraic sum of the moments of the forces about either of the co-ordinate axes has a finite value not zero; for, this gives a resultant of zero magnitude acting at an infinite distance, as $\bar{r} = \frac{\Sigma Pr}{\Sigma P} = \infty$. This is called an indeterminate form; for, as the moment has a finite value the arm cannot be infinite. The condition indicates that the effect is that of a couple, for the sum of the forces forming a couple is zero and the moment of a couple always has a finite value not zero. Moreover, if we assume that one of the forces of the couple passes through the point chosen for the origin, the moment of that force will be zero and the moment of the force of the couple *not* at the origin is finite and equal to the moment of the couple. Therefore, when the algebraic sum of a number of parallel forces is zero and the sum of the moments of the forces about any point

has a finite value not zero, the form $\frac{\Sigma Pr}{\Sigma P}$ is indeterminate and the resultant of the forces is a couple. If we assume that one of these forces passes through the origin, the moment of the other force of the couple about the origin is equal to ΣPr .

24. Case of Three Parallel, Non-Coplanar Forces.—Let three non-coplanar, parallel forces, p_1 , p_2 and p_3 , act in the same direction, perpendicular to the plane of the paper, at the points A , B and C , Fig. 14.

The resultant of p_1 and p_2 will act at the point D , so that $p_1 : p_2 :: DB : DA$ (Art. 21). The resultant of p_3 with this force ($p_1 + p_2$) will act at E , so that

$$p_3 : p_1 + p_2 :: ED : EC.$$

By taking the forces in any other order, the resultant will be found to pass through the same point E .

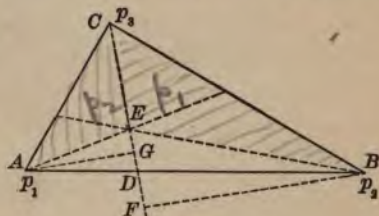


FIG. 14.

Drawing lines from the vertices to this point E , the large triangle is divided into three triangles whose areas are proportional to the three forces, the area of the large triangle being proportional to the resultant; for, the triangles EAC and ECB , having the common base EC , are proportional to their altitudes, AG and BF , which are proportional to AD and DB (similar triangles). But AD and DB are proportional to p_2 and p_1 by construction. Hence the areas EAC and ECB are proportional to the forces p_2 and p_1 . The triangle EAB may in a similar way be shown to be proportional to p_3 . The resultant, which is the sum, $p_1 + p_2 + p_3$, is obviously proportional to the large triangle ABC .

If one of the forces is reversed the point E will fall *outside* the triangle ABC , but the proposition will still hold true, as it will also if *two* of the forces are *equal*, but opposite in direction

(a couple). The student should prove the latter cases and also that in which the sum of the three forces is zero, in which case the resultant becomes a couple.

The proportions proved in this article make it possible to resolve any force into three parallel components whose lines of action are parallel to the given force and pass through any three points not in the same straight line.

The point in a plane through which the resultant of a number of parallel forces passes is called the *center of parallel forces*.

25. Centers of Pressure. Centers of Gravity.—When a number of parallel forces act in the same direction their resultant passes through a point which may be called the *center of force*. If these forces are indefinite in number over an area, such as the wind pressure on a sail, or water pressure against a vertical wall, the point at which their resultant acts is called the *center of pressure*. In the case of the sail where the pressure is uniform over the area, the center of pressure coincides with the center of gravity of the area of the sail. *only then.*

In practical mechanics the bodies, areas, etc., considered are usually of some symmetrical shape, and with our knowledge of the position of the center of gravity of the more common areas, we can find the center of gravity of these symmetrical areas by the following methods:

- (a) Find the center of gravity of the area of dimensions shown in Fig. 15. This is the cross-sectional area of a steel beam, which being symmetrical has its c. g. lying on the middle vertical line. We have therefore only to find its position on this line.

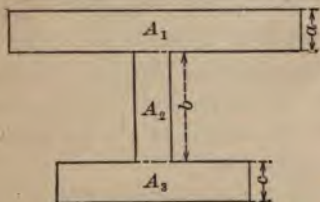


FIG. 15.

Dividing the area into rectangles whose areas are A_1 , A_2 , and A_3 , as indicated, and know-

ing that the c. g. of a rectangle is at its center, if we take moments

about the line representing the lower face of the bottom flange, the sum of the moments of the rectangles must equal the moment of the whole area about that line.

$$A_1(b+c+a/2) + A_2(c+b/2) + A_3(c/2) = (A_1 + A_2 + A_3)\bar{y}.$$

Solving for $\bar{y}$ will give the distance of the c. g. about the lower face of the bottom flange.

(b) Find the center of uniform wind pressure on the sail shown in Fig. 16. We know the c. g. of a triangle is $1/3$ of the altitude up from the base. Therefore, dividing the area as shown, and taking moments about the mast for the abscissa and about the boom for the ordinate of the center of pressure, we have

$$A_1(2a/3) + A_2(a/2) + A_3\left(a + \frac{c}{3}\right) = (A_1 + A_2 + A_3)\bar{x},$$

and

$$A_1\left(b + \frac{d}{3}\right) + A_2(b/2) + A_3\left(\frac{d+b}{3}\right) = (A_1 + A_2 + A_3)\bar{y}.$$

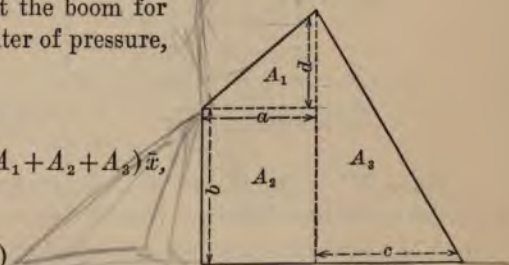


FIG. 16.

Solving the above equations for $\bar{x}$ and $\bar{y}$ will give the abscissa and the ordinate of the center of the uniform wind pressure.

Centers of gravity in general have been fully discussed in the Calculus and are therefore omitted here (see Appendix). *Sl. 124-136*

relative to bodies on surfaces of revolution. p. 228.
26. Theorems of Pappus.—These theorems, having been fully explained in the Calculus, are inserted here merely for reference.

(a) The surface generated by an arc of a plane curve revolving about an axis in its plane, but not cutting the arc, is equal to the product of the length of the arc and the length of the path described by the arc's center of gravity.

This means not only a complete revolution but any fraction thereof.

(b) *The volume generated by a plane area revolving about an axis in its plane but not crossing the area, is equal to the product of the area and the length of the path described by the area's center of gravity.*

27. Problem.—The following non-coplanar, parallel forces act perpendicular to the XY plane at the points whose co-ordinates are given: 4 lbs. at $(2, 3)$; 2 lbs. at $(3, -4)$; and -6 lbs. (a force acting in the opposite direction to the others) at $(-4, 2)$. Find the resultant. Proceeding in accordance with Art. 23:

Force.	Moments about Y .	Moments about X .
4	8	-12
2	6	8
-6	24	12
$\Sigma P = 0$	$\Sigma Px = 38$	$\Sigma Py = 8$

Since $\Sigma P = 0$, by the method of Art. 23, both the $\bar{x}$ and $\bar{y}$ co-ordinates become infinite. Now as the sum of the moments ΣPx and ΣPy , are finite, we must conclude that the resultant is not a single force but a couple. If we assume that one of the forces of this couple passes through the origin, the remaining force of the couple will have the moments, ΣPx and ΣPy , about the co-ordinate axes, or, if F is the magnitude of one of the equal forces of the couple and one of them passes through the origin, the other must have moments about the axes of Y and X , such that

$$Fx = 38 \text{ and } Fy = 8$$

squaring and adding

$$F^2(x^2 + y^2) = 1508,$$

but the square root of $(x^2 + y^2)$ is equal to r , the distance of this force F from the origin, hence $\sqrt{1508} = 38.6(-)$ is the measure of the moment of the couple.

Problems III.

1. Forces of 4 and 9 lbs., respectively, have parallel lines of action 12 inches apart. Find the magnitude and the position of the line of action of their resultant (1st) when they have the same direction and (2d) when they have opposite directions.

2. Five parallel, co-planar forces, 470, 200, 400, 300 and 430 lbs. are acting, the latter four at distances of 5, 10, 12 and 20 ft. from the first force. Find the line of action and magnitude of the resultant.

3. Four parallel, non-coplanar forces of 1, 3, 5 and 9 lbs. act at the following points respectively: $(1, -2)$, $(-4, 1)$, $(4, -2)$ and $(5, 3)$. What is the resultant and where does it act?

4. Resolve a force of 40 lbs. into three parallel forces which will act at the vertices of an equilateral triangle, so that the distances of the resolved forces from the line of action of the original force will be $\sqrt{3}/4$, $\sqrt{7}/4$, $\sqrt{7}/4$ ft.

5. Find the center of pressure of a sail, supposed flat, whose hoist on the mast is 12 ft., which extends 12 ft. along the boom, and whose peak is 15 ft. above the boom and directly over its middle point.

6. Find the center of gravity of a trapezoid whose parallel sides are 8 and 12 ft. long and whose altitude is 5 ft.

7. Parallel, non-coplanar forces of 3, -2, 4 and -5 lbs. act at points whose co-ordinates are $(4, -2)$, $(3, 5)$, $(-1, 6)$ and $(-2, -3)$. Find their resultant and its line of action.

8. Find the volume of a sphere by Pappus' theorem.

9. Find the surface of a ring of circular section whose internal diameter is 1 ft. and whose thickness is 3 inches.

10. Parallel, co-planar forces of 3, 4, 5, 6 and 7 lbs. act at distances of 1, 2, 3, 4 and 5 ft. from a point. If the forces act in the same direction what single force will balance them and where must it act? If the 2d and 4th forces are reversed what single force will balance them and where must it act?

non.1.

CHAPTER IV.

EQUILIBRIUM.

Virtual Work, p. 240
Sl. 136-152

28. *Statics* is that part of Mechanics which treats of the equilibrium of bodies at rest; or, the conditions which govern the forces when rest is produced.

Statical equilibrium is a state of rest produced by the mutual counteraction of two or more forces.

Statical equilibrium is of three kinds: stable, unstable and neutral. A body in equilibrium, which, if slightly moved, *tends to return* to its position of rest, is in *stable equilibrium*.

A body in equilibrium, which, if slightly moved, *tends to move further*, is in *unstable equilibrium*.

A body in equilibrium, which, after being slightly moved, *remains at rest* in the new position, is in *neutral equilibrium*.

As an example, suppose a smooth wire, with a ring on it, is bent into a circle and held in the vertical plane. The ring may be placed in equilibrium at either end of a vertical diameter. When in equilibrium *at the top* of the vertical diameter, any slight motion will displace it, and it will *not* return to its position of equilibrium at the upper end of the vertical diameter where it was in *unstable equilibrium*. But when it is in equilibrium at the bottom end of the vertical diameter, it *will*, after any slight motion, return to rest in its position at the bottom end of the diameter; in this case it was in *stable equilibrium*. Now if the circle, with the ring on it, be placed in the horizontal plane, the ring will be in equilibrium, at any point on the wire at which it is placed and will remain, after any slight change of position, at rest in its new position; it is therefore in *neutral equilibrium*.

Again, when a body is in *unstable* equilibrium, any slight motion will lower its center of gravity, therefore, its center of gravity is at its highest point; or, as we will find later, its potential energy is at a maximum.* When in *stable* equilibrium, any slight motion of the body will raise its center of gravity, therefore, its center of gravity is at its lowest point; or, its potential energy is at a minimum.* When a body is in neutral equilibrium, no slight motion of the body will change the level of its center of gravity; or, its potential energy has a constant value.

29. Equilibrium of a Particle.—A particle may be defined as a minute portion of matter; or, it is a weighted point (a point being without dimensions). The forces which act on a particle must then be limited to forces which *act at a point*. If such forces produce rest it is obvious that their resultant must be equal to zero, for if the resultant were not zero it would cause motion and the particle would no longer be at rest. The *condition* then for the equilibrium of a particle is that the resultant of all the forces acting on it shall be equal to zero. This condition is satisfied—

(a) *For forces all in one plane*, if, having resolved all the forces along and perpendicular to *any* straight line in the plane, we find the algebraic sum of the resolved parts, both along the line and perpendicular to it, equal to zero; or, resolving all the forces along the rectangular axes of X and Y , equilibrium results if the algebraic sums of both the resolved parts along X and those along Y are equal to zero. Briefly, $R^2 = \Sigma \bar{X}^2 + \Sigma \bar{Y}^2$. For equilibrium $R=0$, hence $\Sigma X=0$ and $\Sigma Y=0$.

(b) *For non-coplanar forces*, if the algebraic sums of the resolved parts along *three* rectangular axes are each equal to zero; or, briefly $R^2 = \Sigma \bar{X}^2 + \Sigma \bar{Y}^2 + \Sigma \bar{Z}^2$; for equilibrium, $R=0$, hence ΣX , ΣY and ΣZ must equal zero.

* The reference to potential energy will be understood after studying kinetics.

different { particle body } couples

30. Equilibrium of a Rigid Body.—The forces which may act on a rigid body are *not* necessarily *concurrent*, for the rigid body has dimensions and forces may act at any point of it. As with a particle, a rigid body held at rest under the action of a number of forces, would require the resultant of the forces to be zero. But we have seen that the resultant of a number of forces may be zero and still the forces may be represented by a couple. In such a case the couple would cause motion, therefore it is necessary when considering the equilibrium of a *rigid body* to make certain that no such couple exists. The moment of a couple will *always* have a finite value no matter about what point or axis we take the moment of its forces, therefore, if we take moments, of all the forces acting on a rigid body, about the three co-ordinate axes and find their algebraic sums are equal to zero, we may assume that no couple exists. The *conditions for the equilibrium of a rigid body* then, are, that the resultant of all the forces acting on it must be equal to zero *and* that the algebraic sum of the moments of all the forces, about any point or line, must equal zero. As *any* system of forces may be reduced to a single force, a couple, or a single force and a couple, these conditions are satisfied—

(a) *For forces all in one plane*, if the algebraic sums of the resolved parts of all the forces, both along and perpendicular to *any* straight line in the plane, are equal to zero, *and also*, the algebraic sum of the moments of all the forces acting, about *any* point in the plane, is zero; or, calling the moments about any point H , equilibrium results when

$$\Sigma X=0, \quad \Sigma Y=0 \quad \text{and} \quad \Sigma H=0.$$

Note that $\Sigma H=0$ if $\Sigma H_x=0$ and $\Sigma H_y=0$, the symbols ΣH_x and ΣH_y representing the sums of the moments of the forces about the axes of X and Y respectively.

(b) *For non-coplanar forces*, if the algebraic sums of the resolved parts along *three* rectangular axes are zero *and* the algebraic sum of the moments about *any* line is equal to zero; or, as

the condition that the sum of the moments about *any* line shall be zero, is satisfied if the sum of the moments about the three co-ordinate axes is zero, the conditions for the equilibrium of a rigid body under the action of non-coplanar forces, are, calling the moments about the three co-ordinate axes, H_x , H_y and H_z , respectively,

$$\begin{aligned} \Sigma X &= 0, \quad \Sigma Y = 0, & \Sigma Z &= 0, \\ \Sigma H_x &= 0, \quad \Sigma H_y = 0 & \text{and} \quad \Sigma H_z &= 0. \end{aligned}$$

31. Polygon of Forces. Triangle of Forces.—Referring to Art. 6 we see that the resultant of a number of *concurrent, co-planar* forces, represented by vectors, will be the closing side of a polygon whose other sides are the forces taken in order. When equilibrium exists, the resultant is zero and therefore when in equilibrium, the forces themselves, when represented by vectors, will form a *closed* polygon. In many cases the forces acting on a rigid body (co-planar forces) may be readily reduced to three by finding the resultant of forces whose lines of action intersect. If we can reduce the number of forces to three, the above polygon becomes a triangle, and, if we know one of the forces (and we usually know the weights) and the directions of the other two, we may find the magnitude of the unknown forces as follows: Draw a vector representing the known force and from its ends draw lines in the directions of the two unknown forces. These lines intersecting will form the *triangle of forces*. In this triangle the length of the sides are proportional to the forces in the same ratio as the side representing the known force is proportional to that force.

This method, as described above, depends upon accuracy of construction, but in many cases the sketch of conditions (the force diagram) will furnish a triangle whose sides, being parallel to the lines of action of the forces, are either known or have lengths that are in a known ratio to each other, in which case the proportions are readily stated in the form of an equation. Or, knowing the angles and one side solve by trigonometry.

32. General Method of Solving Problems in Equilibrium.—

Most problems in equilibrium can be solved in several different ways and the shortest way cannot always be selected even after considerable experience. The method of resolving all the forces along the co-ordinate axes will give two, and if the forces are non-coplanar, three equations, from which two or three, as the case is, unknown quantities may be obtained. If other equations are needed, *take moments*; and, after that, form equations from the geometric conditions until there are as many equations as there are unknown quantities, when solve by algebra. The first and most important thing is to draw a figure showing the conditions and see that *all the forces acting are represented* as nearly as possible in the way they act as to direction and with their magnitudes indicated. Then form as many equations as there are unknown quantities and the solution becomes algebraic.

33. Smooth Bodies. Machines.—The smooth body, in a mathematical sense, is one which offers *no frictional resistance* to another body with which it is in contact. A smooth peg or pulley would allow a cord in tension to run *freely* about it without in the least changing the tension of the cord or reducing the motion; again, a smooth inclined plane, though it would constrain a body to slide down its slope, would offer no frictional resistance to its motion.

It follows that the resistance offered by smooth bodies must act only *in the normal* to the point of contact.

In the case of the smooth peg (circular section) the line of action of the resistance would coincide with the radius of the peg at every point at which it was in contact with the cord; and the inclined plane would present a resistance perpendicular to the slope at each point of contact as the body slides down.

Practically, smooth bodies are impossible, but conditions sometimes approach so closely to those of the smooth body that we may, with little error, accept the results obtained under that

assumption; for example, a ladder rests against an oily, horizontal, steel shaft. Scarcely any error would result from assuming that the resistance of the shaft would act normal to the ladder, or, as the ladder rests tangent to the shaft, along the radius produced.

A machine is an arrangement of bodies so connected that motion may be transmitted and force modified either in magnitude, direction, or point of application. Under statics we must consider the motion as just on the point of starting; that is, when the forces are in statical equilibrium. Force, having been applied to a machine, is transmitted to a point where some resistance is to be overcome and finding the relation between the force applied and the resistance overcome is the usual problem. Under statics we may find this relation when equilibrium exists, then any increase in the applied force will start motion.

When the force applied is less than the resistance overcome the machine is said to give us a *mechanical advantage*; when the opposite is the case, we work at a mechanical disadvantage. Of course, in complicated machines, a considerable percentage of the applied force may be lost in overcoming resistances in the machine itself, so that the force which acts on the resistance which it is desired to overcome is only a part of that which is actually applied to the machine. We shall not as yet consider this part of the problem, but confine ourselves to simple machines in which the loss in the machine may be neglected. There are six simple machines: the *lever*, the *wheel and axle*, the *inclined plane*, the *wedge*, the *screw* and the *pulley*.

34. Problems.—(a) A force P holds a small body, of weight W , at rest on a smooth plane which is inclined at an angle a to the horizon. What is the magnitude of P if it is acting along and up the plane?

By adhering to the method of Art. 30, we may resolve all the forces *along* and *perpendicular to any* straight line, but it is

usually more convenient when inclined planes are involved, to

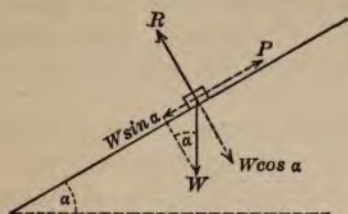


FIG. 17.

resolve *along* and *perpendicular* to the plane. In drawing the diagram of forces *all the forces acting must be indicated*. The resistance of the plane is in the direction of the force R (see Art. 33), the other forces being as indicated. W , being the only

force which is *inclined* to the direction of the plane, is the only one we must resolve; its resolved part *along* the plane is $W \cdot \sin \alpha$, and *perpendicular* to the plane is $W \cdot \cos \alpha$. The former acts down the plane and balances P , while the latter acts perpendicular to the plane and balances R , the body being in equilibrium. We have then the equations:

$$P - W \sin \alpha = 0$$

and

$$R - W \cos \alpha = 0$$

from which the magnitudes of both P and R are readily obtained.

This problem may be solved by the *triangle of forces*, thus:

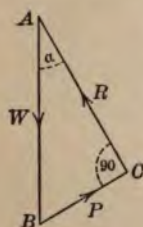


FIG. 18.

Draw AB , representing W , and from its ends BC and AC in the direction of the forces P and R , then the angle BAC is equal to α and knowing that the angle ACB is a right angle, P is equal to $W \sin \alpha$ and R is equal to $W \cos \alpha$, as before; or, by proportion R is to W as the length of AC is to the length of AB , and $P : W :: BC : AB$.

If results are obtained from *measured* sides the greatest care must be taken in constructing the triangle. It is much better to use some other method, unless the force diagram furnishes a triangle of *known dimensions* whose sides are parallel to the lines of action of the *forces*.

It will be observed that the arrow heads follow each other around the sides of the triangle. This is *always* the case in a polygon of *forces in equilibrium*, so if we know the direction of one of the forces we may always find the direction of any of the others by referring to the polygon of forces (see Art. 6).

(b) A weight W is supported by three equal cords l long, which are attached to the ceiling at the vertices of an equilateral triangle, whose sides are equal to half the length of the cords. Find the tension of the cords.

From symmetry the vertical components of the tension in the three cords must be equal and their sum must be equal to the weight W ; also, the weight being at rest, the horizontal components must have a resultant zero.

It is only necessary then to find the vertical component of one of the tensions and to do this we must know the cosine of the angle one of the cords makes with the vertical. Obviously the line of action of the weight W will pass through the center of gravity of the equilateral triangle. This point is $2/3$ of the altitude from a vertex, or,

$$\frac{2}{3} \sqrt{l^2/4 - l^2/16} = l\sqrt{3}/6.$$

The distance from this point to the weight will be

$$\sqrt{l^2 - \left(\frac{l\sqrt{3}}{6}\right)^2} = \frac{l\sqrt{33}}{6}.$$

The required cosine is this vertical distance divided by l , or $\sqrt{33}/6$; hence the vertical component of the tension of one of the cords is $T \cdot \sqrt{33}/6$, and

$$\frac{3T\sqrt{33}}{6} = W,$$

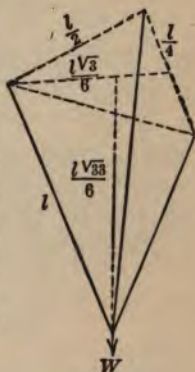


FIG. 19.

which gives

$$T = \frac{2W}{\sqrt{33}}.$$

This solution is much simpler than resolving along three axes and in many cases like advantage may be taken of symmetrical conditions.

(c) Geometrical conditions are of frequent aid; as, a uniform beam of weight W and length b , rests in equilibrium with one end against a smooth vertical wall and the other end held by a cord of length m , which is fast to the end of the beam and to a point in the wall. To find wall resistance and tension of cord.

The line of action of the weight of a rigid body *always* passes vertically through the center of gravity of the body.

The above beam being uniform has its center of gravity at its middle point and, referring to Fig. 20, the line of action of its weight must, from the similar triangles ADC and EGC , bisect the line AC , or m ; therefore, EC is equal to $m/2$.

Three forces in equilibrium, unless they are parallel, *must* meet in a point, for, if they did not, there would be a moment of the third force about the point of intersection of the lines of action of two of them, which is contrary to the condition of equilibrium. The resistance R must therefore pass through E , and it must also be perpendicular to the vertical wall (see Art. 33). Then from the similar triangles ABC and ADE , we have $AD = AB/2$. Calling $AD = DB = y$ and $BC = 2DE = x$, we have from the triangles DCB and ABC ;

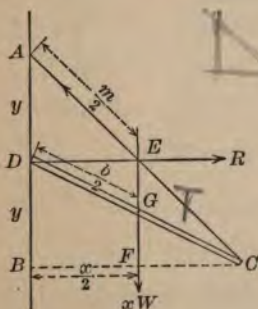


FIG. 20.

$$b^2 = x^2 + y^2 \quad \text{and} \quad m^2 = x^2 + 2y^2$$

from which

$$x = \sqrt{\frac{4b^2 - m^2}{3}} \quad \text{and} \quad y = \sqrt{\frac{m^2 - b^2}{3}}.$$

Having these values any one of the triangles ABC , ADE or EFC will be a triangle of forces. Using ABC , we have

$$R : W :: x : 2y \quad \text{or} \quad R = \frac{x}{2y} \cdot W = \frac{1}{2} \sqrt{\frac{4b^2 - m^2}{m^2 - b^2}} \cdot W,$$

$$T : W :: m : 2y \quad \text{or} \quad T = \frac{m}{2y} \cdot W = \frac{m\sqrt{3}}{2\sqrt{m^2 - b^2}} \cdot W.$$

These results show that m must be limited to a length between b and $2b$, for if $m=b$, both R and T are infinite, and if $m=2b$, R will be equal to zero and T to W .

(d) The uniform lower boom of a ship is b long and of weight W . It is secured to the ship by a smooth hinge and held in a horizontal position by a topping lift which makes an angle of 30° with the vertical and is fast to the boom one-fourth of its length from the outer end. Find the tension of the lift and the amount and direction of the total thrust at the smooth hinge.

Here we have the boom held in equilibrium under the action of three forces. Their lines of action must therefore intersect in the same point, and having the intersection of two of them (the point C where the lines of action of the weight and the tension in the lift intersect), we know the line of action of the total thrust at the smooth hinge must lie in the line AC . (The three forces must meet at a point for equilibrium.)

Having the force diagram we now have the choice of several methods to solve the problem:

(1) *Resolution of Forces*.—We will resolve along and perpendicular to the boom. By geometry we find the distances as indicated in Fig. 21, and from them the functions of the angle θ .

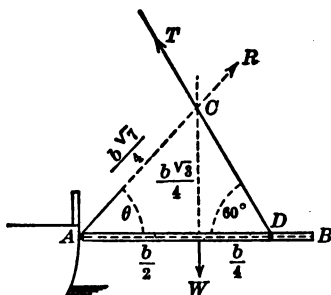


FIG. 21.

Resolving horizontally

$$R \cdot \cos \theta - T \cdot \cos 60^\circ = 0$$

or

$$\frac{2R}{\sqrt{7}} - \frac{T}{2} = 0.$$

Resolving vertically

$$R \cdot \sin \theta + T \cdot \sin 60^\circ - W = 0$$

or

$$\frac{R\sqrt{3}}{\sqrt{7}} + \frac{T\sqrt{3}}{2} - W = 0$$

from which equations

$$R = \frac{\sqrt{7}}{3\sqrt{3}} \cdot W \quad \text{and} \quad T = \frac{4}{3\sqrt{3}} \cdot W.$$

(2) *By Taking Moments.*—If we take moments about the point A, Fig. 21, we eliminate the thrust R, for it has no moment about A as it passes through A. The perpendicular from A upon the line of action of T is $\frac{3}{4} \cdot b \sin 60^\circ = \frac{3\sqrt{3}}{8} \cdot b$; therefore,

$$T \cdot \frac{3\sqrt{3}}{8} \cdot b - W \cdot \frac{b}{2} = 0,$$

$$T = \frac{4}{3\sqrt{3}} \cdot W.$$

Taking moments about D to eliminate T, the perpendicular from D upon the line of action of R is $\frac{3}{4} \cdot b \sin \theta = \frac{3\sqrt{3}}{4\sqrt{7}} \cdot b$; therefore,

$$R \cdot \frac{3\sqrt{3}}{4\sqrt{7}} \cdot b - W \cdot \frac{b}{4} = 0,$$

$$R = \frac{\sqrt{7}}{3\sqrt{3}} \cdot W.$$

(3) By the *triangle of forces*. Not recommended, as figure shows no triangle of *known* dimensions whose sides are parallel to the lines of action of the forces, and, as the triangle will be oblique it would be a longer method than either of the others.

Problems IV.

1. A weight of 100 lbs. is supported by two ropes, one of which is horizontal and the other makes an angle of 60° with the horizontal. Find the tension of each rope.

2. A uniform spanker boom, 32 ft. long and weighing 400 lbs. is topped up to an angle of 30° with the horizontal by a lift made fast to its outer end and which is at an angle of 60° with the boom. If the hinge at the mast is smooth find the tension of the lift, the total thrust at the mast and the angle this thrust makes with the mast.

3. A spar used as a derrick is supported at 30° with the vertical by two equal backstays, the angle between which is 45° and whose plane makes an angle of 60° with the vertical. Find the thrust on the spar and the tension of each backstay when 10 tons is hung from the derrick head.

4. A uniform ladder weighing 40 lbs. rests at an angle of 45° between a smooth vertical wall and a smooth floor. A man who weighs 160 lbs. is $\frac{3}{4}$ of the way up the ladder. If the bottom end is prevented from slipping by a horizontal cord which is fast to the wall, what is the tension of the cord?

5. A beam 8 ft. long and weighing 100 lbs. is suspended from a hook by two cords, 6 and 10 ft. long, respectively, attached to its ends. Find the angle the beam makes with the vertical and the tension of each cord.

6. A crane is formed of a vertical post fixed in the ground; the part AB , above the ground being 15 ft. long, the horizontal member BD being 12 ft. long and the strut is jointed to the post at A and to the horizontal member at C , a point 8 ft. distant from B . If 10 tons hangs from D , find the forces acting at C and B .

whose distance from wall is b

7. A uniform heavy rod, length $2a$, rests in equilibrium over a smooth horizontal rail and with one end against a smooth vertical wall. Find the angle the rod makes with the vertical.

8. A body weighing 10 lbs. rests on a smooth inclined plane with a force of 2 lbs. acting up the plane and a force of 5 lbs. acting horizontally. Find the inclination of the plane.

9. An iron bar weighs 10 oz. per inch. It rests in equilibrium on a table with 3 ft. 3 in. projecting over the edge and a weight of 50 lbs. on the end on the table. How long is the rod?

10. A beam 30 ft. long balances at a point 10 ft. from one end. If 10 lbs. is hung from the other end it balances at a point 2 ft. from the original point. Find the weight of the beam.

200. 8
CHAPTER V.Sl. 178-231
FRICTION.

35. Friction.—Until now we have assumed all bodies to be *so smooth* as to offer none but a normal resistance to other bodies with which they are in contact. We all know, as a matter of *fact*, that this assumption is not true. The surfaces of bodies are *never* perfectly smooth and no two surfaces are exactly alike no matter how carefully they have been prepared. Clearly then if two surfaces are in contact, the little inequalities will cause a resistance to motion if we attempt to slide the surfaces on each other, and this resistance will be increased if the surfaces are *pressed* together.

This resistance to motion between two rubbing surfaces is called the *frictional resistance* or simply *friction*, and it is proportional to the *normal* (normal to the surfaces in contact) pressure between the surfaces, and obviously acts to retard any motion and hence acts in the direction opposite to that in which the motion starts.

This resistance is a *force* opposing any tendency to motion and it must be considered in all practical problems. If the force with which we attempt to drag a body over a rough surface is too small, the body will not move. *Because* it does not move, we know that the resistance of friction *balances* the force (the pull) used; if we slowly increase the pull we shall finally get one which *just starts* the body, and if the line of action of this force (the pull) lies in the *line of the motion* it will just exceed the frictional resistance between the two surfaces. This assumes no other than the frictional resistance to be acting to oppose the motion. As any such force which is not sufficient to move a body is always balanced by the frictional resistance, the *limit* of statical resistance is reached when the body is just on the point of moving. Therefore,

the force which, acting in the line of the tendency to motion, just fails to drag a body over a rough surface, is a measure of the *limiting statical friction* between the surfaces. If the force used be greater than this limit, the body will move.

Frictional resistance still acts when motion exists and is then called *kinetic friction*.

36. Laws of Friction.—The results of many experiments are laid down in the three laws which follow. The laws were published in the year 1781 by Coulomb as the results of his experiments.

I. *The limiting statical friction for a given pair of surfaces depends upon the nature of the surfaces and is proportional to the normal pressure between them.*

II. *Statical friction acts in the direction opposite to the tendency to motion and is independent of the area of the surfaces in contact.*

III. *Kinetic friction is independent of the speed and its direction is opposite to the motion.*

More recent experiments have developed the fact that the kinetic friction is somewhat less than the limiting statical friction, and becomes still less as the speed increases. This is due to the air which gets between the surfaces and acts as a lubricant and also to the smoothing effect of the rubbing the surfaces get. However, the results found by accepting the third law will be very close to the truth for low speeds; that is, the ordinary practical speeds. For high speeds the value of μ in the table below is a trifle large.

Calling the limiting statical friction F and the *normal* resistance R , we have by the first two laws that F varies as R , or

$$F = \mu R,$$

μ being a constant which is equal to the ratio of the frictional resistance to the normal pressure and which is called the *coefficient of friction*.

for instance 100 lbs weight for a pair of surfaces - 20 lbs force
force = 40 lbs

The following table will give an idea of the range of values of μ for different materials:

Wood on wood	μ varies from	.3	to	.7
Wood on metal	"	"	.3	" .6
Metal on metal	"	"	.15	" .24
Metal on metal (lubricated)	"	"	.05	" .08
Stone on stone	"	"	.4	" .6
Rope (hemp) on wood	"	"	.5	" .8
Leather on metal	"	"	.4	" .6
Leather belting	"	"	.25	" 1.00

say $\frac{1}{5}$

37. Angle of Friction.—The angle of friction is the angle the line of action of the total resistance (that is, the vector sum of the *normal* and *frictional* resistances) of a rough surface makes with the *normal* to that surface and its tangent is equal to the coefficient of friction; for, if a body, resting on a rough horizontal surface *AB*, Fig. 22, is just on the point of moving under the action of a horizontal force *P*, the resistance to motion will be equal to μW , as the *normal* resistance, *R*, is equal to *W* when the body rests on a *horizontal* plane, and the *force employed to cause motion is also horizontal*. This resistance, or friction, will act in the direction indicated, opposite to the tendency to motion. The *total* resistance is clearly the resultant of the normal resistance, *R*, and the friction, *F*. By definition the angle of friction is α and its tangent is equal to the friction divided by the normal resistance, or, $\frac{\mu W}{W}$, hence

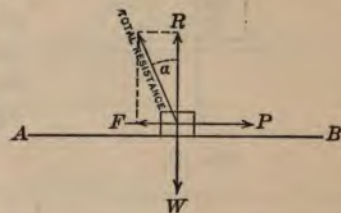


FIG. 22.

$$\tan \alpha = \mu.$$

Here we may note a curious thing about friction. The normal resistance, R , is constant, under the conditions shown in Fig. 22, but we may vary the force, P , from zero, to a magnitude equal to μR , without causing the body to move (that is, without disturbing the equilibrium), therefore the friction, acting in the opposite direction to P , must vary through the same limits as we increase P , being always of a magnitude just sufficient to balance P . The actual magnitude of the friction in this case is, therefore, equal to the force P whenever the magnitude of P is *less* than the limiting static friction.

38. Least Force to Overcome Friction.—Let a body of weight W , rest on a rough horizontal plane and let a force P , Fig. 23, act

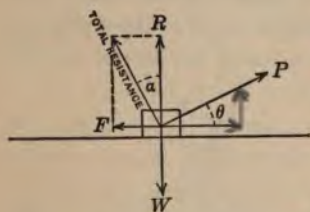


FIG. 23.

in a line making an angle θ with the plane; then if the body is just on the point of moving it is in equilibrium and the sum of the resolved parts of all the forces along and perpendicular to the plane must be equal to zero; therefore, F being equal to μR .

$$P \cdot \cos \theta = F = \mu R, \quad (1)$$

from (1) and (2) we have

$$W - P \sin \theta = R, \quad (2)$$

$$P = \frac{\mu W}{\mu \sin \theta + \cos \theta}. \quad (3)$$

To make P a minimum we must make the denominator of the right member of equation (3) a maximum and, calling the denominator S , and proceeding by the methods of calculus:

$$dS/d\theta = \mu \cos \theta - \sin \theta = 0,$$

from which

$$\tan \theta = \mu = \tan \alpha \quad \text{or} \quad \theta = \tan^{-1} \mu = \alpha,$$

*least value of $P = W \sin \alpha$
instead of $W \tan \alpha$*

which shows that the force P , in order to be the *least force* which will put the body on the point of moving, must act at an angle with the plane such that the coefficient of friction will be its tangent; or, that force which, acting perpendicular to the *total resistance* of the plane, just causes the body to move, will be the least force that can cause motion.

It should be noted that the vertical component of the force P *together* with the normal resistance of the plane are equal to W ; in other words the vertical component of the force P helps to support the weight W .

39. Limits of Equilibrium on a Rough Inclined Plane.—When bodies rest on rough inclined planes the friction helps to keep

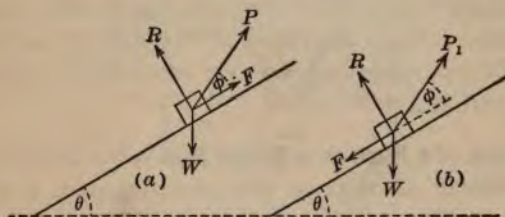


FIG. 24.

them from sliding down; and it will also resist an effort to drag them up. The bodies are in equilibrium when just about to slide down; or, when just on the point of starting when the effort is to drag them up, and the magnitudes of the forces necessary to these two conditions are quite different.

Referring to Fig. 24 (a), the body is just about to slide *down* the plane, therefore friction, which is equal to μR , acts *up* the plane. Resolving along and perpendicular to the plane;

$$P \cdot \cos \phi + \mu R = W \cdot \sin \theta, \quad (1)$$

and

$$P \cdot \sin \phi + R = W \cdot \cos \theta, \quad (2)$$

from which

$$P = \frac{W(\sin \theta - \mu \cdot \cos \theta)}{\cos \phi - \mu \cdot \sin \phi}. \quad (3)$$

Referring to Fig. 24 (b), the body is just about to be dragged *up* the plane, hence the friction (equal to μR) acts *down* the plane. Resolving along and perpendicular to the plane as before,

$$P_1 \cdot \cos \phi - \mu R = W \cdot \sin \theta, \quad (4)$$

and

$$P_1 \cdot \sin \phi + R = W \cdot \cos \theta, \quad (5)$$

from which

$$P_1 = \frac{W(\sin \theta + \mu \cdot \cos \theta)}{\cos \phi + \mu \cdot \sin \phi}. \quad (6)$$

For any force acting at this angle with the inclined plane, whose magnitude lies between the values of P and P_1 (equations 3 and 6), the body will remain at rest; or, the body is in equilibrium (see latter part of Art. 37).

40. Tension of a Rope on a Rough Post.—We all know that if a sufficient number of turns be taken round a post, a rope under any tension (within its strength) may be held at rest. This is of course due to the friction of the rope on the post, which acts in the direction opposite to the tendency to slipping caused by the tension. This friction, at *any point* at which the rope is in contact with the post, is equal to the normal resistance of the post at that point multiplied by the coefficient of friction. Now force, being a vector quantity, includes direction as one of its properties, and to change its direction continually, as does a rope in tension wrapped on a post, requires the action of forces other than the tensions acting along the axis of the rope.

Referring to Art. 5, if we represent the tensions at any two points by vectors, the line joining their ends will represent the mean difference between the tensions at the two points. Referring to Fig. 25, we will take two points indefinitely near to each

other, and resolve the difference between the tensions along and perpendicular to the tangent. The tangential component will be equal to the difference, dT , of the magnitudes of the tensions at the ends of the arc ds , and, as the rope is at rest, this must be balanced by the friction on this arc. The component of the tension difference which is *perpendicular* to the tangent, is the force which causes the change of *direction* of the tension, and it must be balanced by the oppositely directed resistance offered by the post. The angle between the tensions at the ends of the arc ds is $d\phi$, and as the tangents at the ends of the arc are perpendicular to

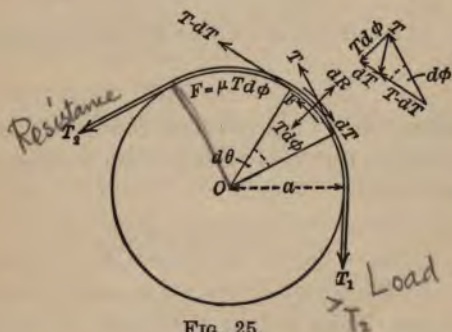


FIG. 25.

the radii to its ends, $d\phi$ is equal to $d\theta$. Also, if T is the tension at one end of the arc ds , the radial component of the tension difference will be $Td\phi$. This radial component is equal to the normal resistance of the post, hence the friction is equal to $\mu T \cdot d\phi$, and the arc ds will be in equilibrium under the action of the forces dT and $\mu T \cdot d\phi$ which act along the tangent and neutralize each other; and the force $T \cdot d\phi$ and the resistance of the post, dR , which act radially and neutralize each other. We have then, substituting for $d\phi$ its equal $d\theta$, the angle at the center of the post,

$$dT = \mu T \cdot d\theta \quad \text{or} \quad \frac{dT}{T} = \mu d\theta,$$

integrating

$$\log T \Big|_{T_2}^{T_1} = \mu \theta \Big|_0^\theta,$$

or

$$\log_e T_1 - \log_e T_2 = \mu \theta \quad \text{or} \quad T_1 = T_2 e^{\mu \theta}, \quad (1)$$

which equation is readily solved by the use of logarithms, the formula for use with common logarithms being

$$\log_{10} T_1 = \log_{10} T_2 + .4343 \mu \theta. \quad (2)$$

Note that in (1) and (2) the angle θ is in circular measure and $e = 2.7182818$.

41. Problems.—(a) Find the least force which will just keep a body weighing 30 lbs. from sliding down a plane of inclination 30° if μ is equal to .2.

In this case friction will act *up* the plane and the line of action of the *total resistance* due to the plane will lie at an angle whose tangent is μ with the normal to the plane, and the least force which will keep the body from sliding down must act perpendicular to this line, or at an angle $\tan^{-1}(-\mu)$ with the plane (prove by making P of equation 3, Art. 39, a minimum). Therefore by substituting $\tan^{-1}(-\mu)$ for ϕ in (3) Art. 39, we have

$$P = \frac{15(5 - \sqrt{3})}{\sqrt{26}} = 9.7 \text{ lbs.}$$

(b) A uniform beam of weight W and length b , rests in the vertical plane between a rough horizontal floor and a rough vertical wall. If the coefficient of friction is in both cases μ find the angle the beam will make with the floor when just on the point of slipping.

If the beam slips, its upper end will move down and its foot

will move away from the wall, therefore friction will act *upward* on the wall and *toward the wall* on the floor. Referring to Fig. 26, the total resistance of the floor, R , will be the resultant of the normal resistance, N , and the friction μN , and will act at an angle α , the tangent of which is μ , with N ; also S , the total resistance of the wall, is the resultant of M and μM , and acts at an angle α (μ being equal on the floor and wall) with M . The beam, being just on the point of slipping, is in equilibrium under the action of the three forces W , R and S , and these three forces must meet in the point C , and making the angle at C between R and S equal to a right angle, for the lines of action of R and S both make an angle α with the normals N and M . Resolving R and S back into their rectangular components, we have

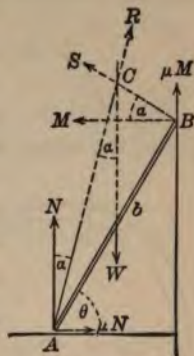


FIG. 26.

$$N + \mu M = W, \quad (1)$$

and

$$M = \mu N, \quad (2)$$

from (1) and (2) we have

$$M = \frac{\mu W}{1 + \mu^2} \quad \text{and} \quad N = \frac{W}{1 + \mu^2} \quad (3)$$

Taking moments about A

$$M \cdot b \cdot \sin \theta + \mu \cdot M \cdot b \cdot \cos \theta = W \cdot \frac{b}{2} \cdot \cos \theta, \quad (4)$$

substituting the value of M given by (3) in (4), and solving for θ , we get

$$\tan \theta = \frac{1 - \mu^2}{2\mu};$$

and the complete solution gives

$$S = \frac{\mu W}{\sqrt{1 + \mu^2}} \quad \text{and} \quad R = \frac{W}{\sqrt{1 + \mu^2}}.$$

(c) Find the friction on a trunnion seat.

Let the trunnion support a weight W . When the trunnion is at

rest it is in equilibrium under this weight acting downward and the equal resistance of the trunnion seat which acts upward, at A (Fig. 27). The seat being a little larger (exaggerated in the figure) than the trunnion, if the trunnion turns clockwise, it will, because of the friction, roll up the seat, as shown in the figure, until the friction becomes equal to the component, $R \cdot \sin \theta$,* which tends to make it slide down.

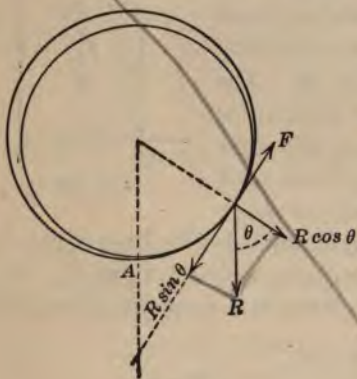


FIG. 27.

These forces will act in oppo-

site directions along the tangent to the trunnion seat, at the line of contact between the trunnion and its seat.

The friction, F , is equal to $\mu \cdot R \cdot \cos \theta$, therefore,

$$\mu \cdot R \cdot \cos \theta = R \cdot \sin \theta, \text{ or } \tan \theta = \mu.$$

* R is not necessarily equal to the weight supported by the trunnion seat, but is a combination of the weight and the force which causes the trunnion to turn. In fact R is the resultant of all the forces acting on the trunnion seat and it may not act directly downward. The angle θ , assuming R is in a plane perpendicular to the axis of the trunnions, is the angle the line of action of R makes with trunnion radius to the point of contact. The above will cover conditions such as those involved when a heavy horizontal shaft is turned by a symmetrical couple.

As R is the resultant of the friction, or its equal $R \cdot \sin \theta$, and the normal resistance of the trunnion seat, or $R \cdot \cos \theta$, we have

$$R^2 = F^2 + (R \cdot \cos \theta)^2,$$

or

$$F = R \sqrt{1 - \cos^2 \theta} = R \cdot \sin \theta = \frac{\mu R}{\sqrt{1 + \mu^2}} = R \sin \phi.$$

The normal resistance of the trunnion seat is equal to $\frac{R}{\sqrt{1 + \mu^2}}$.

Problems V.

1. A 500 lb. weight rests on a horizontal plane for which the coefficient of friction is .2; what force must be applied to a rope to move it: (1) if the rope is horizontal? (2) if the rope makes an angle of 45° with the horizontal? (3) what is the least force?

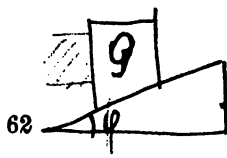
2. A uniform ladder stands between a horizontal floor and a vertical wall. μ is $\frac{1}{4}$ at each end. What is the least angle the ladder can make with the floor?

3. If it takes three times as great a force to drag a body up a rough inclined plane as it does to keep the body from sliding down, find the coefficient of friction. Forces act parallel to the plane.

4. A uniform rod leans against a smooth vertical wall making an angle of 30° with it. The lower end is on a rough floor and is about to slide; what is the coefficient of friction?

5. If the angle of friction is 15° , what is the least force that will pull a weight of 200 lbs. up a plane of 45° inclination? In what direction does it act?

6. A 100 lb. weight rests on a rough plane ($\mu = \frac{1}{3}$) inclined at 30° with the horizontal and is just kept from sliding by a force P , acting along the plane. What is the value of P ? What is the least force that will start the weight up the plane, and its direction?



if the only friction is bet. surfaces
 I need to get from that value of

MECHANICS.

P just sufficient to raise G ; $P = 9 \tan \theta$

7. A ladder inclines 45° between a smooth wall and a rough ($\mu = \frac{1}{4}$) pavement. How far up may a man, who weighs six times as much as the ladder, climb?

8. If $\mu = \frac{1}{4}$, and a man can pull 100 lbs. what tension on a rope could he hold by taking three turns round a post?

9. A ship's hawser can stand a tension of 10,000 lbs. How many turns round the bitts must a man who can pull 250 lbs. take to hold the hawser till it breaks? $\mu = .168$.

10. A horizontal shaft bearing supports three tons. If the coefficient of friction is .12 what couple would turn the shaft if its arm is one foot?

Chapt. Va nov. 15

Review of fundamental formulae of Calculus.

Chapt. Vb nov 22

Review of fundamental properties of curves

circle
 ellipse
 parabola
 hyperbola
 Sine curve
 cycloid

Dec 6
Sl. 163-177
CHAPTER VI.

SYSTEMS OF RIGID BODIES. SUSPENSION BRIDGE. CATENARY.

42. Equilibrium of a System of Rigid Bodies.—When a number of rigid bodies are connected they form a *system* and each separate rigid body is a *member* of the system.

A *system* of rigid bodies is in *statical equilibrium* when *every member* of the system remains at rest under the action of *all* the forces to which it is subjected.

To many systems of rigid bodies separate names have been given, such as derricks, sheer-legs, bridge and roof trusses, etc., and when investigating their equilibrium, we may consider the conditions for the equilibrium (1) of the system as a rigid whole, (2) of each rigid member, separately, and (3) of any particular point or particle.

The conditions for the equilibrium of (1) and (2) are those for a rigid body, *i. e.*, that the resultant of all the forces acting shall equal zero *and* that the algebraic sum of the moments, about any point or line, of all the forces acting, shall equal zero. These conditions are satisfied: (a) *the forces being co-planar*, if the algebraic sums of the *X* and of the *Y* components of all the forces acting are each equal to zero *and* the algebraic sum of the moments of all the forces, about any point in the plane, is equal to zero; (b) *the forces being non-coplanar*, if the algebraic sums of the *X*, *Y* and *Z* components of all the forces are each equal to zero *and* the algebraic sums of the moments of all the forces acting, about the *X*, *Y* and *Z* axes, are each equal to zero. The condition for the equilibrium of (3) is that the resultant of all the forces acting on the particle (at the point) shall equal zero.

43. Stress.—Heretofore all forces that have been considered under the subject of equilibrium have been such as are known as *external forces*, that is, *outside forces* which act *on* a body. But when the equilibrium of a particular *particle* in a system (or member) is considered we must include *all* forces acting on *that particle* and some of these forces will be the reactions of other particles on the one selected. Such reactions * are the stress—tension or compression—in the member itself, and such forces are generally known as *internal forces*, though they are external to the particle in question.

Stress is the name given to the forces which act among and through the particles which make up a rigid body, and the body is said to be *in tension* or *in compression* in accordance with the tendency these forces have to *stretch* or *compress* the body.

Members in tension exert a pull, from each end toward their middle, on members connected to their ends; and those in compression exert a push, from their middle toward their ends, on members connected to their ends. Thus a member having its ends held fast by other members of the system, would, if under a tension of T lbs., exert a pull of T lbs. toward its middle point, on the members at *each* of its ends; and, one under a compression of C lbs. would exert a push of C lbs. on the members at each of its ends.

These pushes or pulls that a member exerts on its neighbors (due to the *internal forces*), must be taken into account when considering the equilibrium of a member which is a part of a system of rigid bodies.

44. Equilibrium of Points.—A system of rigid bodies is not necessarily rigid in itself; as, a number of bars whose ends are joined to each other by a single, smooth, loose rivet, will assume a position of equilibrium if the end bars are secured to *any* two

* These reactions and their effects on the bodies in which they act form a special branch of Mechanics called Strength of Material.

points. In considering such a system *as a whole* we would take into account the weight of the whole system, and the two forces which are the reactions at the points of support, these being the only *external* forces acting. If considering *one member*, the external forces would be, the weight of the member and the reactions at its ends due to the two adjoining members. If considering *a point in the member*, its equilibrium is established by the push or pull, which acts in both directions and thus neutralizes itself, of the compression or tension which is in the member; but if we choose a point at the end, say *between two members*, let us take the smooth rivet above which connects two members, we then have to consider the push or pull of the internal stress of *both* the members which are connected at this point, *in addition to* any other external forces at that point.

Thus the *important points* are those *at the ends* of the members, as all points between the ends are obviously in equilibrium, being subject only to the internal stress which balances itself. If then we investigate *all the joints* and also the *points of support* of the system, we will obtain complete information as to its equilibrium.

45. Weight of Members.—When considering the equilibrium of the joints we must not neglect the weight of the members. This always acts at the center of gravity of the member, but it is *supported* at whatever points the member is supported; therefore, the weight must be resolved into parallel, vertical components, whose lines of action pass through the points of support of the member.

In the case of uniform members (in practical structures the members are generally uniform) the weight acts at the middle point and if the member is supported at the ends, half the weight may be said to act at each end.

46. The Funicular Polygon.—Let five equal bars, each of weight W and length a , be connected to each other by a single smooth rivet, and let the system be supported at two points, distance apart b and at the same level (see Fig. 28). Considering

the system as a whole, the external forces will be the weight, $5W$, whose line of action passes vertically through the middle of the middle bar, and the two resistances of the supports S . The whole system being in equilibrium under the action of these three forces, the triangle (a) Fig. 28, will be the triangle of forces representing its equilibrium.

The joint (1) at the left support will be in equilibrium under

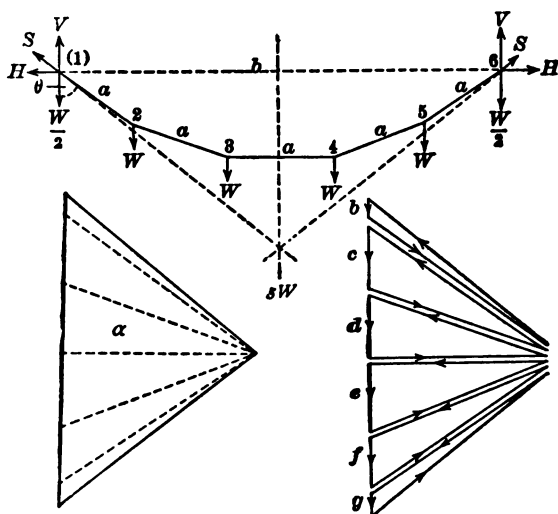


FIG. 28.

the action of the resistance of the support, S , the tension of the first rod and $W/2$, the part of the weight of the first rod supported at that point. The triangle of forces representing its equilibrium is (b) Fig. 28.

The joint (2) is in equilibrium under the tensions of the first and second rods and the weight W at the joint. Triangle of forces (c).

The joints (3), (4) and (5) and (6) being represented by the triangles (d), (e), (f) and (g).

It will be observed that the six smaller triangles, if put together so that the lines representing the weights form a vertical line, will make a triangle which is equal in all respects to the triangle of forces (*a*) which represents the equilibrium of the whole system. This figure represents in addition to the three forces which hold the system in equilibrium, the tension of each of the bars forming the system. Also by referring to the small triangles the *direction* of the stress on any joint may be ascertained.

Figure (*a*), with the dotted lines included, is called the Funicular Polygon. From this polygon, if we know the weights and the angle θ , we may calculate the condition of stress of any of the members of the system.

If the resistances of the supports, S , S , be resolved into their horizontal and vertical components, H and V , it is clear that the horizontal component will be equal to the horizontal component of the stress in the first member; also, that the vertical component will be equal to half the whole weight, or $5W/2$; the resultant of this vertical support and the $W/2$ from the first rod, which acts down at this point, is $2W$; the actual tension of the first rod at its middle is the resultant of H and this $2W$, and acts in the direction along the rod. This being the case, if the rod were shortened indefinitely, so that the part of its weight, which acts downward at the support, would become infinitesimal, the direction of the supporting resistance, S , would act in the direction of the first rod prolonged. In other words, if the system consisted of a heavy, flexible cord (a chain of innumerable links) the supporting resistance would be tangent to the cord at the supports.

47. The Suspension Bridge.—Suspension bridges are built by hanging a platform from cables stretched between two piers. In these bridges the number of tension rods (rods by which the roadway or platform is hung from the cables) is very great, so that we may, with little error, consider the cable to be a continuous curve;

is equal to y , and BC is equal to $x/2$; therefore,

$$\frac{H}{x/2} = \frac{wx}{y}, \quad \text{or} \quad y = \frac{w}{2H} \cdot x^2. \quad (1)$$

To obtain the value of H , take moments about A ,

$$b \cdot H = (wa) \cdot a/2, \quad \text{or} \quad H = \frac{wa^2}{2b}.$$

Substituting in (1)

$$y = \frac{b}{a^2} \cdot x^2. \quad (2)$$

Equation (2) is that of a parabola with a vertical axis and a latus-rectum equal to a^2/b .

To find S , take moments about O . The perpendicular from O on the line of action of S , is, $p = \frac{a}{2} \sin \theta$. But $\tan \theta = \frac{2b}{a}$, hence

$$\sin \theta = \frac{2b}{\sqrt{4b^2 + a^2}}; \text{ therefore,}$$

$$S \cdot \frac{ab}{\sqrt{4b^2 + a^2}} = wa \cdot \frac{a}{2}, \quad \text{or} \quad S = \frac{wa\sqrt{4b^2 + a^2}}{2b}. \quad (3)$$

S is the resistance of the pier and is also the tension of the cable at the pier. This value of the tension is readily shown to be the greatest. H is its least value.

In the above formulæ, b is known as the "dip" and a as the "half span."

The tension at any point, P , is

$$T = \frac{w}{2b} \sqrt{4b^2x^2 + a^4} = \frac{H}{a^2} \sqrt{4b^2x^2 + a^4}.$$

48. The Catenary.—A uniformly heavy cord suspended from fixed points by its ends, hangs in a curve called the catenary. The weight in this case is uniform *along the arc* and therefore the exact position (the abscissa) of its line of action with regard to any assumed vertical axis is often unknown.

Let Fig. 30 represent a perfectly flexible, uniformly heavy cord, suspended from the points A and B . As in the preceding article that part of the cord from O' to B , is in equilibrium under the action of the three forces, H at O' , L at B , and the weight, acting at the center of gravity of the part $O'B$, of the cord, and, with sufficient data, we could use the triangle of forces to obtain the values of H and L . But as we do not know the abscissa of the line of action of the weight, we will solve the problem in another way. We will take the lowest point, O' , of the cord as origin. At any point, P , whose co-ordinates are x and y , we will

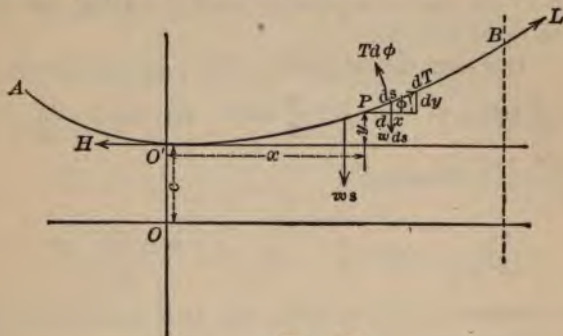


FIG. 30.

have, as in Art. 40, the radial component of the difference between the tensions at the ends of an arc ds long, equal to $T \cdot d\phi$, and the arc, ds , will be in equilibrium under the action of $T \cdot d\phi$, which acts along the radius of curvature, dT , [$dT = (T + dT) - T$, the difference of the tensions at the ends of the arc], which acts along the tangent at the point, and $w \cdot ds$ (the weight), which acts vertically downward through the middle of the length ds (see Fig. 30). Resolving these forces along and perpendicular to the tangent at P , we have

$$dT = w \cdot ds \cdot \sin \phi, \quad (1)$$

$$T \cdot d\phi = w \cdot ds \cdot \cos \phi, \quad (2)$$

substituting $\sin \phi = dy/ds$, and integrating (1)

$$T = wy + \left[k = H \right]_{\substack{\phi=0 \\ T=H \\ \therefore k=H}} \quad \text{or} \quad T = wy + H. \quad (3)$$

Dividing (1) by (2) we have

$$dT/T = \tan \phi \cdot d\phi. \quad (4)$$

Integrating, $\log T = -\log \cos \phi + k$ or $T \cdot \cos \phi = \text{a constant}$.

When $\phi = 0$, $T = H$, hence

$$T \cdot \cos \phi = H, \quad \text{or} \quad T \cdot dx = H \cdot ds. \quad (5)$$

Equation (5) divided by (2) gives

$$\frac{w}{H} dx = \sec \phi d\phi. \quad \text{ie. expression which is in the form of length}$$

If in (6) we let $H = wc$, so that H is equal to the weight of a length c of the cord, we get by integrating

$$\frac{x}{c} = \log(\sec \phi + \tan \phi) + [C = 0],$$

or

$$\frac{dy}{dx} + \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = e^{\frac{x}{c}}. \quad (7)$$

solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{1}{2} \left\{ e^{\frac{x}{c}} - e^{-\frac{x}{c}} \right\} \quad (8)$$

and integrating

$$y = \frac{c}{2} \left\{ e^{\frac{x}{c}} + e^{-\frac{x}{c}} \right\} + k. \quad (9)$$

(9) is the equation of the catenary with the origin at the lowest point of the curve. Here we know that when $x=0$, $y=0$; and substituting we get $0=c+k$, therefore k (the constant of integration) must be equal to $-c$. Transferring the origin from O' to O ,

making the distance $O'O$ equal to c , we will have

$$y' = \frac{c}{2} \left\{ e^{\frac{x}{c}} + e^{-\frac{x}{c}} \right\}, \quad (10)$$

which is the equation of the catenary in most general use. The horizontal line through O is the directrix.

49. The method of the preceding article may be used to deduce the equation for the curve of the suspension bridge cables. Remembering that the weight on the length ds of the cable is, in this case, $w \cdot dx$, we have, by resolving the forces which act on ds along and perpendicular to the axis of X (the forces are $w dx$, $T d\phi$ and dT). Resolving vertically,

$$dT \cdot \sin \phi + T \cdot \cos \phi \cdot d\phi = w \cdot dx. \quad (1)$$

Resolving horizontally,

$$dT \cdot \cos \phi - T \cdot d\phi \cdot \sin \phi = 0. \quad (2)$$

From (2)

$$dT = T \cdot \tan \phi \cdot d\phi.$$

Which substituted makes (1)

$$\sec \phi \cdot d\phi = \frac{w}{T} \cdot dx. \quad (3)$$

Integrating (2) we have

$$\log T = -\log \cos \phi + k,$$

or

$$T \cdot \cos \phi = \text{a constant} \Big|_{T=H}^{\phi=0} \quad \text{or} \quad T \cdot \cos \phi = H,$$

substituting $T = H \cdot \sec \phi$ in (3), integrating, and putting $\frac{dy}{dx}$ for $\tan \phi$, we have

$$\frac{dy}{dx} = \frac{w}{H} \cdot x,$$

integrating

$$y = \frac{wx^2}{2H} + [k=0], \quad (4)$$

and substituting $H = \frac{wa^2}{2b}$ from Art. 47, we have, as before, the equation,

$$y = \frac{b}{a^2} \cdot x^2. \quad (5)$$

50. Problems.—(a) A spar used as a derrick is supported at an angle of 30° with the vertical by two equal backstays the angle

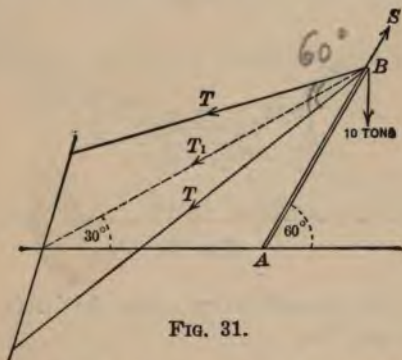


FIG. 31.

between which is 60° and whose plane makes an angle of 60° with the vertical. Find the thrust in the spar and the tension of each backstay when 10 tons hangs from the derrick head. From symmetry the tension of the two backstays will be equal, and they may be replaced by a single backstay, in the same plane as the spar and the 10 ton weight (see Fig. 31), which will be under a tension,

$$T_1 = 2T \cdot \cos 30 = T\sqrt{3}.$$

Considering the equilibrium of the point B, we have resolving horizontally and vertically

$$T_1 \cdot \cos 30^\circ = S \cdot \cos 60^\circ, \quad (1)$$

$$T_1 \cdot \sin 30^\circ + 10 = S \cdot \sin 60^\circ. \quad (2)$$

From (1) $S = T_1 \sqrt{3}$, which substituted in (2) gives

$$T_1 = 10 \text{ tons. Hence } T = 10/\sqrt{3}. \quad S = 10\sqrt{3}.$$

(b) If a cord, fixed at its ends, carries weights, W_1, W_2, W_3 , etc., at intervals, show that the horizontal components of the tensions, T_1, T_2, T_3 , etc. (see Fig. 32), are constant; and, that if the weights are equal, the tangents of the angles the consecutive parts of the cord make with the horizontal, form an arithmetical series.

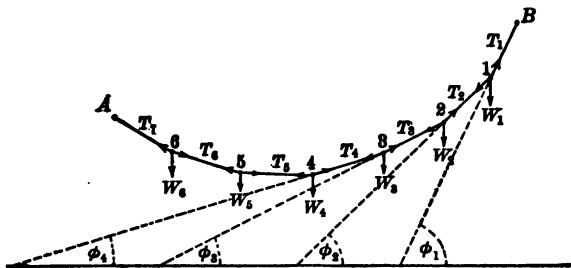


FIG. 32.

The joint (1) is in equilibrium under the forces T_1, W_1 and T_2 . Resolving Hor. and Ver.

$$T_1 \cos \phi_1 - T_2 \cos \phi_2 = 0, \quad (1)$$

$$T_1 \sin \phi_1 - W_1 - T_2 \sin \phi_2 = 0. \quad (2)$$

The joint (2) is in equilibrium under the forces T_2, W_2 and T_3 . Resolving Hor. and Ver.

$$T_2 \cos \phi_2 - T_3 \cos \phi_3 = 0, \quad (3)$$

$$T_2 \sin \phi_2 - W_2 - T_3 \sin \phi_3 = 0. \quad (4)$$

(1) and (3) show that $T_1 \cos \phi_1 = T_2 \cos \phi_2 = T_3 \cos \phi_3 = \dots$, etc., or the horizontal components are equal to a constant. Calling this H ,

$$T_1 = H \sec \phi_1. \quad T_2 = H \sec \phi_2 \dots, \text{ etc.,}$$

and (2) and (4) become

$$H \tan \phi_1 - W_1 - H \tan \phi_2 = 0, \quad (5)$$

$$H \tan \phi_2 - W_2 - H \tan \phi_3 = 0. \quad (6)$$

From (5) and (6) we have

$$\tan \phi_1 - \tan \phi_2 = W_1/H,$$

$$\tan \phi_2 - \tan \phi_3 = W_2/H, \text{ etc.}$$

From which, the W 's being assumed equal, the difference between any two consecutive tangents is a constant and the tangents are in arithmetical series.

(c) Show that the length of arc of the catenary is equal to

$$s = \frac{c}{2} \left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}} \right).$$

For any curve $ds = \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \cdot dx$ and from equation (8),

Art. 48,

$$\frac{dy}{dx} = \frac{1}{2} \left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}} \right),$$

hence

$$ds = \sqrt{1 + \frac{1}{4} \left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}} \right)^2} \cdot dx,$$

reducing

$$ds = \frac{1}{2} \left(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}} \right) \cdot dx,$$

integrating

$$s = \frac{c}{2} \left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}} \right).$$

Also; combining this value of s with equation (9), Art. 48, remembering that $k = -c$, gives

$$(y + c)^2 = s^2 + c^2, \quad \text{or} \quad s^2 = y^2 + 2cy.$$

For this equation the origin is at the lowest point of the curve.

Problems VI.

1. Fourteen tons hang from a tripod whose legs are equal and inclined at 30° to the vertical. Find the thrust in each leg.

2. The lower ends of a pair of sheer-legs of 40 ft. vertical height, are 20 ft. apart; the upper ends being 12 ft. out horizontally from the vertical plane through the lower ends, and the backstay is fixed on the ground at a distance of 60 ft. from the vertical plane through the feet of the sheers. Find the tension of the backstay and the thrust in each leg when the sheers support a weight of 30 tons.

3. A flexible cord is suspended from two points at the same level. Four equal weights are hung from the cord so as to divide the span into five equal parts, a . If the horizontal part of the cord is a distance d below the supports find the horizontal pull on the supports. What are the angles made by the different parts of the cord?

4. A cord, hung between two points, supports three weights, of 10, 12 and 20 lbs., respectively, if one of the segments of the cord is horizontal, and under a tension of 16 lbs., find the angles the other segments make with the horizontal and their tensions.

5. The two rafters which support a roof are 15 and 20 ft. long, respectively, and the tie rod is 25 ft. long. If 1750 lbs. are to be supported at the peak, find the stress in the rafters and tie rod.

6. A suspension bridge weighs $833\frac{1}{3}$ lbs. per foot of roadway. If the span is 192 ft. and the dip 15 ft. what is the greatest and least tension of the cables?

7. If the load on one of the cables of a suspension bridge is $\frac{1}{2}$ ton per foot of span, and the span is 600 ft. and the cable dips $\frac{1}{13}$ of the span, find the greatest and least tension of the cable. Find also the whole pressure on the pier if the stays make the same angle with the pier as do the cables.

8. A chain hung from two point at the same level and 80 ft. apart, sags 20 ft. If the horizontal tension is equal to the weight of a length of 43 ft. of the chain, how long is the chain? p. 76

9. Prove that the tension at any point of the catenary is equal to the weight of a part of the cord whose length is equal to the ordinate of the point (origin at O , Fig. 30).

$$T = w/y + c$$

~~Chapt. VI_a Dec. 13~~
~~Review of Fundamental Formulas of Strength of Materials~~

Chapt VI_a Dec. 20 13
 Elements of diff. eqns.

KINEMATICS. *Dec 20*CHAPTER VII. *11. 7-23*

RECTILINEAR MOTION.

51. Definitions.—Kinematics treats of *pure motion*; that is, motion without reference to the mass of the thing moved or to the forces which cause the moving. In kinematics we are interested only in the relation between the *space* traversed and the *time* required to traverse it. This chapter will be restricted to the consideration of straight line (rectilinear) motion and the next one to curvilinear motion.

When a point moves along a straight line its position at any instant is known if we know its distance at the instant from a fixed point on that line. Such a fixed point is called *the origin*. If we measure the time from the instant the point is at the origin, the ratio between the space traversed, s , and the time, t , will be the *speed* of the motion, which may be defined as *the space traversed in one unit of time*. This is based upon the assumption that the speed remains constant; that is, that equal spaces are traversed in equal time intervals.

We must not confuse the meaning of this word *speed* with the meaning of the word *velocity*. In a mathematical sense, velocity means more than speed. Velocity includes the ideas both of *space traversed in unit time* and the *direction* of the motion. Thus *velocity* is a *vector* quantity, while *speed* is a *scalar* quantity. Velocity is the magnitude of the rate of change of position *together with* the direction of the motion, while speed is only the magnitude of the rate of change of position. When considering the motion of a point we must know both its speed and the direc-

tion of the motion and the word velocity, in mathematics, is understood to embrace both of these ideas.

The units of space and time are the foot and the second.

Kinematics is the investigation of pure motion; or, the relation between the time and the space traversed by a moving point.

Speed is the rate of the change of position; or, the length of path traversed in a unit of time.

Velocity is the direction of the motion together with the speed.

52. Uniform Velocity.—When a point, traveling in a fixed direction, moves so that, at any part of its journey, it passes over the same amount of space in the same interval of time, it is said to be moving at a *constant* or *uniform* velocity. The magnitude of such a velocity is equal to the quotient of any portion of the space traversed divided by the number of units of time required to traverse that portion of space; or,

$$v = \frac{s}{t}. \quad (1)$$

Thus knowing the fixed direction in which the point is traveling and having the magnitude of the motion from (1), we have the velocity fully described. It will be noted that the space in the above equation may be as great or as small as desired, but it is necessary that the time be that time required in traversing the selected space. We may therefore write the equation, $v = \frac{\Delta s}{\Delta t}$; or, passing to the limit,

$$v = \frac{ds}{dt}. \quad (2)$$

53. Varying Velocity.—Velocity may vary either by changing the direction of the motion (curvilinear motion), or by changing the magnitude of the speed. When investigating rectilinear motion we are confined to the latter variation.

When a point does *not* traverse the same amount of space in the same time interval at different parts of its journey, it is said to

have a varying velocity. When this is the case, equation (2) of the preceding article will give us the magnitude of the velocity (the speed) at any point in the path, and equation (1) will give us the *mean* magnitude of the velocity during the time interval, t .

The magnitude of the velocity given by equation (2) is the space which would have been traversed in a unit of time if, from the instant under consideration, the point had continued on with a *uniform* velocity.

54. Acceleration.—When a point has a varying velocity the rate of change of the velocity, either in speed or direction or both, is called the *acceleration*. Rectilinear acceleration is *positive* or *negative* as the speed is increasing or decreasing. A negative acceleration is called *retardation*. Acceleration may be uniform or varying. When the acceleration is uniform the speed of a point increases or decreases by equal amounts in equal times. Uniform acceleration is measured by the *change of velocity* in a *unit of time*; or, if a is the acceleration, v_1 the velocity at the beginning of an interval t , and v_2 the velocity at the end of the interval, then

$$a = \frac{v_2 - v_1}{t}.$$

When the acceleration is varying the mean acceleration for any interval of time, t , will be given by the above equation, and by indefinitely diminishing the interval of time, the change of velocity will be Δv (v_2 above is equal to $v_1 + \Delta v$), and $a = \Delta v / \Delta t$, or passing to the limit

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \quad (1)$$

which will give the acceleration whether the rate of the velocity is constant or varying.

When the acceleration is constant we get by integrating equation (1)

$$v = at + C,$$

C , the constant of integration, being the value of v when t is zero. Calling this value V_0 , we have

$$v = V_0 + at = ds/dt, \quad (2)$$

which determines the velocity.

By integrating equation (2), we have

$$s = V_0 t + \frac{at^2}{2} + C,$$

the constant of integration being the value of the space when t is zero. Calling this S_0 , we have for the position of the moving point after any elapsed time t ,

$$s = S_0 + V_0 t + \frac{at^2}{2}. \quad (3)$$

55. Acceleration of Gravity.—The most familiar example of a varying velocity and a constant acceleration is given by a body falling freely from rest. It has been proved by experiment that bodies falling in this way increase their velocity by 32.2 feet per second during every second of their fall. This increase of velocity of 32.2 feet per second (written 32.2 f/s²) during each second of the fall is called the *acceleration of gravity* and is usually designated as g . It is constant for any place on the earth's surface, but changes a little with a change of latitude of the observer.

If it is desired to find the velocity of a point, in terms of its initial velocity, after it has traversed a space s , we have by squaring both sides of equation (2), Art. 54, and assuming $s=0$ when $t=0$,

$$v^2 = V_0^2 + 2a \left(V_0 t + \frac{at^2}{2} \right) = V_0^2 + 2as. \quad (1)$$

If we substitute the acceleration of gravity, g , for a , and h for the space traversed, we get

$$v^2 = V_0^2 + 2gh, \quad (2)$$

which will give the velocity, after falling through a distance h , of a body projected downward with an initial velocity of V_0 . If the body is projected *upward* with a velocity V_0 , the acceleration of gravity will act downward and the body will be retarded. It is then necessary to make the sign of g negative as the speed is decreasing. The acceleration of gravity, g , may be substituted for a in any of the above formulæ, care being taken to use its proper sign. When bodies fall *from rest*, V_0 in formulæ (1) and (2) is equal to zero, and (2) becomes $v^2 = 2gh$ or $v = \sqrt{2gh}$, which will give the velocity, after falling h feet, of a body simply dropped from rest.

56. Units.—The standard units employed in calculations involving velocities and accelerations are the *foot* and *second*. The velocity of a point given as a feet per second (written, a f/s) means that in one second the point with this velocity will move over a space of a feet, provided the velocity continues unchanged during the whole of the second. Velocities are often given in other units, as miles per hour (written m/h).

When a point with varying velocity increases or diminishes that velocity at the rate of a feet each second, it is said to have an acceleration of a feet per second *per second* (written a f/s²); that is, its velocity each second is a foot-seconds per second greater or less than it was the preceding second.

In working with the preceding and other formulæ it is obviously necessary that *units of the same kind* be used. That is, if velocities are given in miles per hour they must be reduced to feet per second before substituting in the formulæ.

57. Problems.—(a) A point uniformly accelerated from rest attains a speed of 50 m/h in 4 minutes. Find the acceleration in foot and second units.

4 minutes is 240 seconds;

50 m/h is 50×5280 divided by 60×60 or $220/3$ f/s;

substituting in formula (2), Art. 54, remembering that as the point starts from rest V_0 is equal to zero,

$$v = at, \text{ or } 220/3 = 240a; \text{ hence } a = 11/36 \text{ f/s}^2.$$

(b) A point, moving with a velocity of 2000 f/s, is brought to rest in traversing a space of 20 feet. Regarding the retardation as uniform, find its value and the time required to bring the point to rest.

Using formula (1), Art. 52, and the *mean* velocity, 1000 f/s, we find the time required in this way,

$$v = s/t \text{ or } 1000 = 20/t, \text{ from which } t = 1/50 \text{ second.}$$

Having the time, substitute in formula (3), Art. 54, S_0 being zero;

$$s = S_0 + V_0 t + at^2/2, \text{ or } 20 = \frac{2000}{50} + \frac{a}{2 \times 50 \times 50},$$

from which

$$a = -100000 \text{ f/s}^2.$$

The negative sign shows retardation.

(c) With what velocity will a stone, dropped from rest, reach the bottom of a shaft 100 feet deep?

Here we require the velocity after falling through a distance h , as given by the formula $v = \sqrt{2gh}$; and substituting, we find

$$v = \sqrt{2 \times 32 \times 100},$$

or

$$v = 80 \text{ f/s.}$$

(d) A is 160 feet above B . At the instant A drops a body, B projects one vertically upward with a velocity of 80 f/s. In how long and where will the bodies meet?

A 's body will fall for the same length of time that B 's ascends; calling the space A 's body traverses, s_1 , and that that B 's body

traverses, s_2 , $s_1 + s_2$ will equal 160 feet. $s_1 = gt^2/2$ and $s_2 = V_0t - gt^2/2$, so that

$$160 = gt^2/2 + V_0t - gt^2/2, \quad \text{or} \quad 160 = V_0t = 80t,$$

from which

$$t = 2 \text{ seconds};$$

substituting this value of t in the formula for the fall of A 's body, or in that for the ascent of B 's, will give the distance down from A , or the distance up from B , of the point where the bodies meet. The first gives

$$s_1 = gt^2/2 = (32.2 \times 4)/2 = 64.4 \text{ ft. below } A.$$

The second gives

$$s_2 = V_0t - gt^2/2 = 80 \times 2 - (32.2 \times 4)/2 = 95.6 \text{ ft. above } B.$$

58. Vector Representation.—Velocity being a vector quantity, the velocity of a moving point may be definitely expressed by a straight line of as many units in length as the velocity to be represented and drawn in a direction parallel to the motion of the point.

To represent the motion of a point traveling at the velocity of a f/s in a direction making the angle θ with a fixed direction, let BC , Fig. 33, lie in the fixed direction. Lay off accurately the angle θ and draw the line BD . On this line lay off a equal divisions to represent the magnitude of the velocity. If a point at B has a velocity of a of these divisions per second in the direction making the angle θ with OX it would be at D at the end of one second.

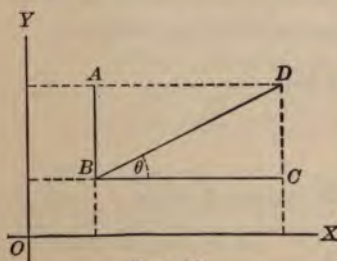


FIG. 33.

59. Component Velocities.—Suppose a point at B , Fig. 33, to be acted upon *simultaneously* by two velocities, one, of magnitude BC and parallel to OX , and the other of magnitude BA perpendicular to OX . At the end of one second the point will be at D , as before, and will have traversed as before the line BD . Thus it is seen that the two velocities BA and BC acting together have accomplished the same result as the original velocity, BD , and have done this in the same time as was required by BD . Thus the velocity BD may be called the *resultant* velocity of the *component* velocities BA and BC . *This is true for either vector or scalar*

60. Composition of Velocities.—Having given two component velocities the resultant velocity may be found by careful construction, as in the figure, or by the ordinary rules of trigonometry. *linear motion*

When the component velocities are at right angles to each other the magnitude of the resultant is given by taking the square root of the sum of the squares; and the tangent of the angle (θ in Fig. 33) which the resultant makes with the X component is the Y component (BA , or its equal, CD , Fig. 33) divided by the X component (BC in Fig. 33).

If the components are *not* perpendicular to each other, as in Fig. 34, the resultant may be found as before by careful construction and measurement, or by the rules of trigonometry, thus, the side BD of the triangle BDC is found by means of the formula

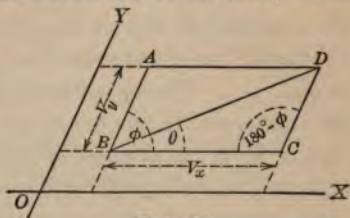


FIG. 34.

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2\overline{BC} \cdot \overline{CD} \cos DCB.$$

Calling the components V_x and V_y and the angle between them ϕ , noting that the angle DCB is equal to $180 - \phi$, we have for the resultant:

$$R^2 = V_x^2 + V_y^2 + 2V_x V_y \cdot \cos \phi, \quad (1)$$

which gives the magnitude; the angle θ is found by the formula

$$\tan \theta = \frac{V_y \sin \phi}{V_x + V_y \cos \phi}. \quad (2)$$

61. Resolution of Velocities.—Given the resultant to find the rectangular components. Referring to Fig. 33, the X component, BC , is equal to the projection of the resultant, BD , upon the fixed line OX ; its magnitude must therefore be equal to $BD \cos \theta$; the Y component is equal to the projection of the resultant upon OY and is therefore equal to $BD \sin \theta$.

To obtain components which do not act at right angles to each other involves the solution of an oblique plane triangle, unless some data, other than the angle between components, is given.

It will be observed that the lines between A , B , C and D , in both of the preceding figures form a parallelogram, and the resultant velocity is the diagonal of this parallelogram, the components being two of the adjacent sides. The figure is called the *parallelogram of velocities*.

62. Polygon of Velocities.—The above parallelogram may be used to find the resultant of a number of velocities as follows: Find the resultant of two of the given velocities, then the resultant of another of the given velocities with this first found resultant, and so on The operation is tedious, but will serve to show that *the resultant of a number of velocities is represented by the closing side of an open polygon of which the other sides are the several velocities taken in order and represented as vectors*. The velocities must be represented by vectors so arranged that the *directions* of the motion shall be consecutive around the polygon; that is, if an arrowhead be placed on the first vector indicating the direction of the motion represented by it, that arrowhead can be moved around the whole polygon and will indicate, when on any of the vectors, the direction of the motion represented by that vector. When on the closing side, which represents the resultant, the direction

should be reversed, see Fig. 35. If we add to the five velocities a sixth equal to the resultant in magnitude but *reversed* in direction, a point acted on by all six velocities would remain at rest. See equilibrium of forces, Art. 34, also Art. 6.

Letting the lines a, b, c, d, e of Fig. 35 be vectors representing five velocities. Constructing the resultant of a and b we find it is

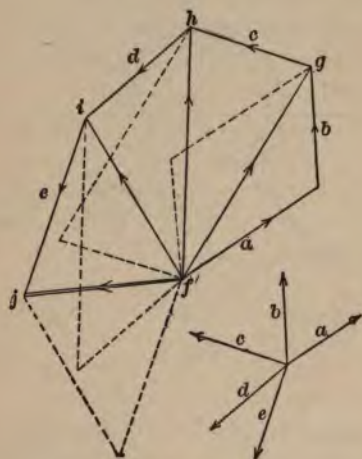


FIG. 35.

represented by fg ; the resultant of fg and c is fh , and so on . . . ; the final resultant being the line fj , which closes the polygon.

63. The foregoing method of finding the resultant of a number of velocities is not recommended, the following being much simpler. It is required to find the resultant velocity and its direction of a point which is subject to the following seven velocities: 5, $2\sqrt{3}$, 6, 8, $5\sqrt{2}$, 16 and $3\sqrt{2}$ f/s acting in directions making respectively the following angles with the first, or 5 f/s velocity; 30° , 60° , 120° , 225° , 300° and 315° (see Fig. 36).

Taking each velocity separately it may be replaced by two component velocities acting at right angles, as in Art. 8. The first velocity has no Y component, being perpendicular to the direction

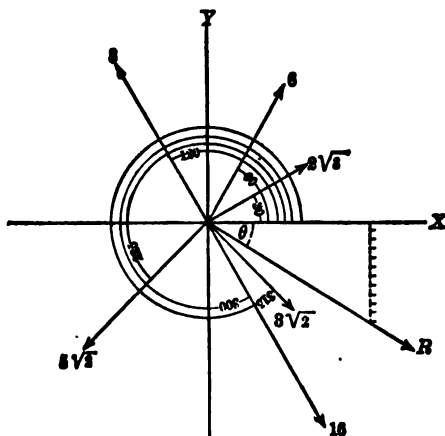


FIG. 36.

X components		Y components	
	5		0
$2\sqrt{3} \cos 30^\circ =$	3	$2\sqrt{3} \sin 30^\circ =$	$\sqrt{3}$
$6 \cos 60^\circ =$	3	$6 \sin 60^\circ =$	$3\sqrt{3}$
$8 \cos 120^\circ =$	-4	$8 \sin 120^\circ =$	$4\sqrt{3}$
$5\sqrt{2} \cos 225^\circ =$	-5	$5\sqrt{2} \sin 225^\circ =$	-5
$16 \cos 300^\circ =$	8	$16 \sin 300^\circ =$	$-8\sqrt{3}$
$3\sqrt{2} \cos 315^\circ =$	3	$3\sqrt{2} \sin 315^\circ =$	-3
ΣX	= 13	ΣY	= - 8

$$R = \sqrt{\Sigma X^2 + \Sigma Y^2} = \sqrt{233}, \quad \tan \theta = \Sigma Y / \Sigma X = -8/13.$$

resolved into
in the proper
on the func-
their resultant

ive the X com-
e sum of the Y
ct. Having both
and by extracting

with the first velocity
component, ΣX , or
e *fourth* quadrant is
at having the negative
This method is again
ethod for working with

leration is also a vector
ding articles for velocities
to accelerations.

arrying velocity in a straight
ed; and the direction of this
me line (rectilinear motion)
be resolved in any direction;
their resultant may be found
cities.

Plane.—Suppose a point at the
at an angle θ to the horizontal,
of gravity. The acceleration of

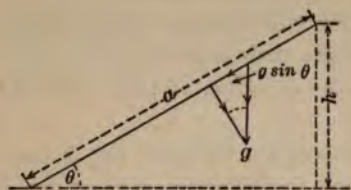


FIG. 37.

gravity, which always acts vertically downward, or toward the center of the earth, may be divided into two components, one of which will act perpendicular to the inclined plane and will have no effect on the motion along the plane; but the other will act along the plane

and cause the point to move down its smooth surface. Referring to Fig. 37, this latter component will be found equal to $g \sin \theta$. Substituting this value for the acceleration in formulæ (2) and (3) of Art. 54, will give us

$$v = V_0 + g \cdot \sin \theta \cdot t, \quad (1)$$

and

$$s = S_0 + V_0 t + \frac{1}{2} \cdot g \cdot \sin \theta \cdot t^2. \quad (2)$$

If we call the vertical height of the plane h and the length of slope a , then $h = a \cdot \sin \theta$. From equation (1) of Art. 55, remembering that the point was at rest at the top of the inclined plane and that therefore V_0 and, the origin being at the top of the plane, S_0 are equal to zero, we have

$$v^2 = 2g \cdot \sin \theta \cdot a. \quad (3)$$

From equation (2) of Art. 55, we have, for the velocity of a point falling freely from the height h , $V^2 = 2gh$. The value of h from the figure is equal to $a \cdot \sin \theta$, which substituted in this formula for the velocity, gives

$$v^2 = 2g \cdot \sin \theta \cdot a. \quad (4)$$

Equations (3) and (4) being identical, show that a body sliding down a smooth inclined plane will have, at the bottom, a velocity equal to that which it would have acquired by falling freely through a space equal to the height of the inclined plane. But

the velocity acquired by sliding down an inclined plane is in the *direction* of the slope of the plane, while that from falling freely is vertical.

66. Relative Velocities.—The rate of change of position of a point *A*, relative to a point *B*, is the relative velocity of *A* with respect to *B*. Either or both of the points may be in motion in any direction. For example, if a point *A* is moving northeast at 40 f/s, and a point *B* is moving east at 30 f/s, the relative velocity of *A* with respect to *B* would be the velocity with which *A* is moving away from *B* at any instant.

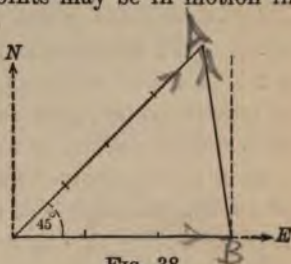


FIG. 38.

If we plot the two velocities as vectors, the straight line connecting their ends (the difference between the vectors) will represent both in magnitude and direction the relative velocity desired (see Fig. 38). The magnitude may be found by accurate plotting and measurement or by formula (1) of Art. 60, remembering to change the sign of the last term, as we want to find not the diagonal *between* the two velocity vectors, but the other diagonal (see Art. 5, equation 2). The formula gives

$$v^2 = 30^2 + 40^2 - 2 \cdot 30 \cdot 40 \cos 45^\circ = 802 \text{ roughly,}$$

$$v = 28 \cdot + \text{f/s,}$$

which shows that *A*, at any instant is moving directly away from *B* at the rate of a little more than 28 f/s in a direction a trifle west of north.

67. Problems VII.

1. A point uniformly accelerated from rest takes 15 seconds to attain a velocity of 15 miles an hour. Find the acceleration.
2. A point is acted upon by velocities of 1, 2 and 3 f/s, re-

spectively, acting at angles of 120° with each other and all in one plane. Find the resultant velocity.

3. On a ship steaming at 12 knots (knot = 6000 ft. per hour) due east the pennant shows a wind from 30° abaft the port beam and the wind gauge registers 20 f/s velocity. Find the true direction and velocity of the wind.

4. Velocities of 4 and 7 f/s make an angle of 30° with each other. Find their resultant.

5. With what velocity must a body, subject to the acceleration of gravity, be projected vertically to rise 100 ft. in 2 seconds and what will its velocity be at that height?

6. A motor car, moving at 30 m/h, is subject to a retardation of 8 f/s². How long will it be in coming to rest and how far will it travel during this time?

7. With what velocity must a point, subject to the acceleration of gravity, be projected up a smooth inclined plane of 100 ft. base and 25 ft. height in order that it may just reach the top?

8. A point, subject to the acceleration of gravity, is projected vertically upward with a velocity of 170 f/s. How much space will it traverse during the third second of its flight?

9. Velocities of 2, 4, 6 and 8 f/s act on a point. They make angles respectively of 30° , 120° , 225° and 270° with a fixed straight line. Find the magnitude of their resultant and its direction with reference to the fixed straight line.

10. The crank of a horizontal engine is one foot long and is making 300 revolutions per minute. When the crank and connecting rod are at right angles the piston has a velocity of 32 f/s. Find the velocity of the end of the crank relative to the piston.

CHAPTER VIII.

CURVILINEAR MOTION.

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68. Path.—When a point is moving, and the motion is confined to a fixed plane, we may know its position at any instant, if we know the distances it was at the instant from any two fixed, intersecting straight lines in the plane. These distances are usually measured along lines parallel to the fixed straight lines, just as, in analytic geometry, the co-ordinates of a point which fix the position of the point are measured along lines parallel to the co-ordinate axes. Thus the position of the moving point at any instant is fixed by the co-ordinates, referred to a fixed origin, of its position at that instant.

If the motion is not confined to a plane, we must know the distances, at any instant (the co-ordinates), of the position from *three* fixed planes. But the similarity to the methods of analytic geometry should be apparent, and we may obviously fix the position of the point at any instant by means of either *rectangular* or *polar* co-ordinates.

The *path* of the point is the locus of the several positions of the point.

The co-ordinates of the position of the point at any instant are *functions of the time* measured from the instant the point was at the initial position.

The *initial position* is the position from which the motion starts; or, it may be any other point in the path that we may select from which to measure time. Thus, when the point is at the initial position, the time is equal to zero.

The *origin* for the co-ordinates of any position in the path is the same as that used in analytic geometry as the origin for the equation of the path.

$x = f(t)$
motion is given by 3 eqns $y = \phi(t)$ eliminate t to give
 $z = \psi(t)$

69. Velocity in Curvilinear Motion.—When the path of a moving point is not a straight line, the velocity of the point is changing, even though the speed remains uniform. The *direction* of the motion changes.

We may find the average (the mean) velocity of the point between any two positions by dividing the shortest distance between the two positions (the displacement) by the time required for the moving point to go from the first to the second position. If we reduce the interval of time indefinitely we will get the mean velocity for an infinitesimal duration of time, for which infinitesimal time the point will be moving in the direction of the tangent to the path. Thus formula (2) of Art. 52, will be true for any motion, and the velocity of a moving point at any position in a curved path will be directed along the tangent to the path at that position and will have a magnitude,

$$v = \frac{ds}{dt}. \quad (1)$$

That is, if the point should continue *for one second*, the motion it has *at the instant considered*, it would move over a space *along the tangent* equal to the *magnitude* of v given by (1) and the motion would be in the *direction* of the tangent to the path.

This is true whether the motion of the point is confined to one plane or not and we may therefore define the velocity of a point moving in a curvilinear path (true for straight path also):

Velocity is the limit of the mean velocity when the time interval is indefinitely diminished (approaches zero).

70. Component Velocities.—Having the velocity at any instant of a moving point along its path, we may separate it into two components which shall be parallel to any two given directions. This may be done by representing the velocity along the path by a vector and resolving it in the two directions as vectors are resolved; or, by calculating the components as indicated in Art. 61. It is always most convenient to resolve parallel to lines which lie

in directions at right angles with each other, and this being the case, the resolutions are usually made parallel to the rectangular co-ordinate axes.

The velocity along the path is called the total velocity and is designated v . The components are called the X and Y component velocities and are designated V_x and V_y .

Of course the component velocities are functions of the time, therefore,

$$V_x = \frac{dx}{dt}, \quad V_y = \frac{dy}{dt}, \quad (1)$$

where dx and dy are component spaces, traversed in time dt , by the projections, on the X and Y axes, of the moving point.

Having the component velocities, the total velocity at any instant is

$$v = \frac{ds}{dt} = \sqrt{V_x^2 + V_y^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \quad (2)$$

Referring to Fig. 39, the total velocity and its component velocities (those two velocities, *parallel to the axes*, which would replace the total velocity), at three points, A , B and C , of the path of a moving point, are shown laid off as vectors. Remember that these vectors represent conditions *at the instant* the moving point was at that particular place, A , B or C , in its path. The magnitude of the total velocity *may not* have been the same at these three points, although in each case it has been called v in the figure. The component velocities obviously vary.

In the above it has been assumed that the moving point *remained in one plane*. Should this not be the case (*i. e.*, for motion in space as it is called), we have simply to resolve the total velocity in directions parallel to the *three* co-ordinate axes, thus getting

$$V_x = \frac{dx}{dt}, \quad V_y = \frac{dy}{dt}, \quad \text{and} \quad V_z = \frac{dz}{dt}. \quad (3)$$

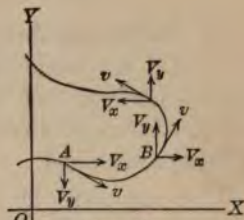


FIG. 39.

Or, having the component velocities in the three directions,

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}. \quad (4)$$

71. Curvilinear Acceleration.—Whenever a point moves in anything but a straight line, the velocity changes; the *speed* may remain uniform, but as the direction of the motion changes the velocity varies its direction and therefore the motion of the point is being accelerated.

If we represent, by vectors, the velocity at two positions of the point in its path, the difference between the vectors will be a vector representing the *mean* change in the velocity during the time taken by the point to travel the path between the two positions. That is, the velocity at the second position, would be the resultant of the velocity at the first position, and some other velocity, which, acting during the time taken by the point to traverse the space along the path between the two positions, would have deflected the velocity to the *direction* and have given it the magnitude it has at the second position. Supposing the velocity at the first position to have been v , at the second position to have been $v + \Delta v$, and the time, which elapsed between the instants the point was occupying the two positions, to have been Δt , then the mean change of velocity *per unit time*, the mean acceleration, would be $\frac{\Delta v}{\Delta t}$; passing to the limit and calling the acceleration a ,

$$a = \frac{dv}{dt}. \quad (1)$$

Thus acceleration may be defined as the limit of the mean acceleration when the time interval is indefinitely diminished.

The above equation is the same as equation (1) of Art. 54, but there, dv meant a change in the velocity which only affected its magnitude, while here, it means a change in the velocity which

affects the *direction* as well as the magnitude. Referring to Fig. 40, A , and B are two positions on the path of a moving point. The velocity of the point when it passed A is represented by the vector V ; when it passed B by the vector V_1 . If we lay off these vectors from a common point C , the vector ΔV , connecting their ends, will represent the mean velocity, both in magni-

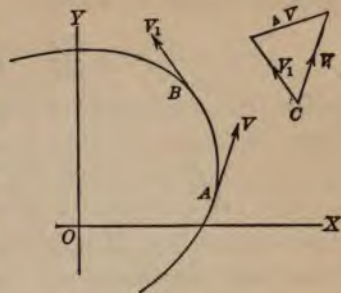


FIG. 40.

tude and direction, that must have acted, together with V , on the point as it traveled from A to B , in order that the velocity of the point should have changed to that represented by the vector V_1 ; for the vector ΔV is the difference between the vectors V_1 and V , and so is a mean value of the velocity which acted with V , during the time Δt , and produced V_1 (see Art. 5).

The value of a (which is called the total acceleration) given by equation (1), is, like that for the velocity given in Art. 69, a value for *the instant considered*. The acceleration at another position in the path would have quite a different direction.

72. Component Accelerations.—Having the direction and magnitude of the total acceleration for any position in the path, we may resolve it along and perpendicular to the tangent to the path at that position. The component along the tangent will be that component which accelerates the speed, while the component perpendicular to the tangent, will be that component which changes the direction of the motion of the point when it is at that position in its path. These components, which are designated a_t and a_n , are frequently useful.

Components parallel to the co-ordinate axes are perhaps most convenient. They are designated a_x and a_y , and remembering the

values of V_x and V_y , we have

$$a_x = \frac{dV_x}{dt} = \frac{d^2x}{dt^2} \quad \text{and} \quad a_y = \frac{dV_y}{dt} = \frac{d^2y}{dt^2}. \quad (1)$$

The total acceleration is found from the component accelerations

$$a = \sqrt{\left(\frac{dV_x}{dt}\right)^2 + \left(\frac{dV_y}{dt}\right)^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}. \quad (2)$$

For motion not all in one plane (motion in space) the *three* co-ordinate components are available as with velocities and may be worked with in the same way.

They are

$$a_x = \frac{dV_x}{dt} = \frac{d^2x}{dt^2}, \quad a_y = \frac{dV_y}{dt} = \frac{d^2y}{dt^2}, \quad a_z = \frac{dV_z}{dt} = \frac{d^2z}{dt^2}. \quad (3)$$

Having the three co-ordinate accelerations, the total acceleration is

$$\begin{aligned} a &= \sqrt{\left(\frac{dV_x}{dt}\right)^2 + \left(\frac{dV_y}{dt}\right)^2 + \left(\frac{dV_z}{dt}\right)^2} \\ &= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}. \end{aligned} \quad (4)$$

73. Trigonometric Conditions.—When we consider *the path* of the moving point, the trigonometric conditions are the same as those in analytic geometry.

Referring to Fig. 41, ds is a portion of the path so short that it may be looked on as coincident with the tangent to the path; dx and dy are the X and Y components of ds . The angle ϕ is the angle ds makes with the X axis and its tangent is equal to $\frac{dy}{dx}$, getting its sign from the *direction of the*

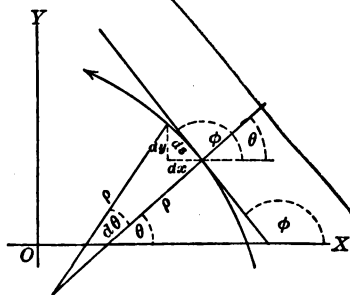


FIG. 41.

motion of the X and Y components of the space traversed; thus in the figure, dx is negative, as the point is moving toward the left at the instant selected. The angle the normal to the direction of the motion makes with the X axis (called θ in Fig. 41) is equal to $\phi - \frac{\pi}{2}$. This angle indicates the direction of the radius of

curvature, ρ , and from the figure $\rho = \frac{ds}{d\theta}$. When ϕ is in the first

quadrant, θ will be negative, and ρ will be directed into the second quadrant. Now referring to Fig. 42, the direction of the mean total acceleration, between two positions on the path where the velocities are V and V_1 , is shown by the vector marked ΔV . The

direction of the X and Y components of this total mean acceleration, are shown by the lines marked ΔV_x and ΔV_y . It is clear that the angle between the X component and the tangent at the position where the velocity is V , is equal to ϕ and that the angle between the Y component and this tangent, is $\phi - \frac{\pi}{2}$; or, that the Y component makes the angle θ (of Fig. 41) with the tangent at the position where the velocity is V .

From the above we have:

$$\tan \phi = \frac{dy}{dx}. \quad \theta = \phi - \frac{\pi}{2}. \quad \sin \theta = -\cos \phi.$$

$$\sin \phi = \frac{dy}{ds}. \quad d\theta = d\phi. \quad \cos \theta = \sin \phi.$$

$$\cos \phi = \frac{dx}{ds}. \quad \rho = \frac{ds}{d\theta} = \frac{ds}{d\phi}. \quad \tan \theta = -\cot \phi.$$

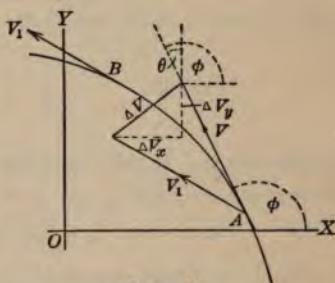


FIG. 42.

74. Tangential and Normal Accelerations.—Having the rectangular components of the total acceleration, we may obtain the

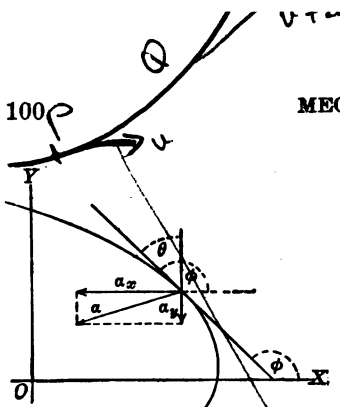


FIG. 43.

a_y , as shown. Resolving these components *along* the tangent, we have

$$a_t = -a_x \sin \theta + a_y \cos \theta = a_x \cos \phi + a_y \sin \phi. \quad (1)$$

Resolving them perpendicular to the tangent, we have

$$a_n = a_x \cos \theta + a_y \sin \theta = a_x \sin \phi - a_y \cos \phi. \quad (2)$$

Multiplying (1) by 2 and substituting for the functions of ϕ and for a_x and a_y , we get,

$$2 \cdot dx \cdot d^2x + 2 \cdot dy \cdot d^2y = 2 \cdot a_t \cdot ds \cdot dt^2. \quad (3)$$

Integrating

$$(dx)^2 + (dy)^2 = (ds)^2 = \int 2 \cdot a_t \cdot ds \cdot dt^2.$$

Differentiating

$$2 \cdot ds \cdot d^2s = 2 \cdot a_t \cdot ds \cdot dt^2,$$

or

$$a_t = \frac{d^2s}{dt^2} = \frac{dv}{dt} = v \frac{dv}{ds}. \quad (4)$$

time der. of v considered f(v)

In like manner, substituting in (2), we have

$$d^2x \cdot dy - d^2y \cdot dx = -(dx)^2 \cdot d\left(\frac{dy}{dx}\right) = ds \cdot a_n \cdot dt^2. \quad (5)$$

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tangential and normal components by resolving the rectangular components along and perpendicular to the tangent at any position in the path.

Referring to Fig. 43, let the total acceleration at any position in the path of a moving point, be represented by the vector a . The X and Y components of the total acceleration will then be a_x and

$$\frac{BD}{\Delta t} = \frac{(v + \Delta v) \cos \Delta \alpha - v}{\Delta t}$$

$$\lim = \frac{dv}{dt}$$

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$$\frac{CD}{\Delta t} = \frac{(v + \Delta v) \sin \Delta \alpha}{\Delta t}$$

$$\lim = v \frac{d\alpha}{dt} = \frac{v^2}{\rho}$$

$$\text{Now } \tan \phi = \frac{dy}{dx}, \text{ from which } d\phi = \frac{d\left(\frac{dy}{dx}\right)}{\sec^2 \phi} = \frac{d\left(\frac{dy}{dx}\right)}{\left(\frac{ds}{dx}\right)^2} \cdot ds = \rho d\alpha$$

$$\text{But } \rho = \frac{ds}{d\phi} = \frac{(ds)^3}{(dx)^2 \cdot d\left(\frac{dy}{dx}\right)}; \text{ or, } (dx)^2 = \frac{(ds)^3}{\rho \cdot d\left(\frac{dy}{dx}\right)}$$

Substituting this value of $(dx)^2$ in (5) we get

$$a_n = - \frac{\left(\frac{ds}{dt}\right)^2}{\rho}$$

Having considered that the positive direction is outward, the negative sign shows that this normal component is directed inward, and neglecting this sign, we have

$$a_n = \frac{\left(\frac{ds}{dt}\right)^2}{\rho} = \frac{v^2}{\rho} \quad (6)$$

In (4) the result has been put in several forms which should be familiar. Also by (4) it is seen that if the speed is uniform, $\frac{d^2s}{dt^2}$ is equal to zero; or, there is *no tangential* acceleration, from which we observe that, if a point is *moving at a uniform speed in a curve*, the whole of the acceleration is employed to change the direction of the motion. Referring to equation (6), we see that, the speed being constant, this effort is constant if ρ is constant; i. e., if the curve is a circle. Again by (6), if the path is a straight line, ρ is infinite and therefore the *normal* component of the total acceleration is zero.

75. Problems.—Suppose a point, subject to the acceleration of gravity only, has a velocity v , in a direction making an angle θ with a fixed horizontal line, at the instant the point crosses the line. Assume the fixed horizontal line to be the X axis and the initial position to be O , Fig. 44. The time will therefore be zero

when the point crosses the X axis. The abscissa of the point after any time t , will be

$$x = v \cos \theta \cdot t. \quad (1)$$

The ordinate, after time t , will be

$$y = v \sin \theta \cdot t - \frac{1}{2}gt^2, \quad (2)$$

for, the point being subject to the acceleration of gravity, will have fallen, during the time t , through a space equal to $\frac{1}{2}gt^2$, so that, instead of being at A , it will be at B , t seconds after being at

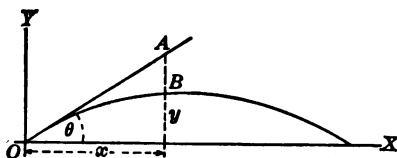


FIG. 44.

O. Eliminating t from equations (1) and (2), the equation of the path is

$$y = x \tan \theta - \frac{gx^2}{2v^2 \cos^2 \theta}, \quad (3)$$

which shows the path to be a parabola. This is known as the "trajectory in vacuo," as it represents the path of a projectile as it would appear if the projectile traveled in a vacuum.

Another way of obtaining this result is as follows: The acceleration of gravity being vertical, and there being no other acceleration,

$$\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = -g. \quad (4)$$

Integrating

$$\frac{dx}{dt} = V_x = 0 + C. \quad \frac{dy}{dt} = V_y = -gt + C_1.$$

The constants of integration, C and C_1 , are the values of the component velocities at the initial position (the origin), therefore

$$V_x = \frac{dx}{dt} = v \cos \theta \quad \text{and} \quad V_y = \frac{dy}{dt} = v \sin \theta - gt. \quad (5)$$

Integrating again

$$x = v \cos \theta \cdot t \quad \text{and} \quad y = v \sin \theta \cdot t - \frac{1}{2}gt^2.$$

Eliminate t , and we have the trajectory as before.

There are a number of interesting things connected with this equation. If we put y equal to zero, and solve for x , we get the extreme horizontal range. Putting $\frac{dy}{dx}$ equal to zero (horizontal

tangent to the path) and solving for x , gives the abscissa of the highest position reached. The student should find the elevation for maximum range and for maximum (highest) position reached.

76. Problem.—Suppose a point to move with a uniform speed along the circumference of a circle of radius a . The speed being uniform, the angle subtended at the center of the circle by that part of the circumference passed over in a unit of time, will be uniform. In other words, the angular velocity will be uniform. Calling this angular velocity ω , the space traversed in a unit of time by the point is $a\omega$, and the angle, θ (Fig. 45), subtended by the arc described by the point in any time t will be ωt . Therefore, the co-ordinates of any position of the point, after a time t from the selected initial position, A , will be

$$x = a \cos \omega t \quad \text{and} \quad y = a \sin \omega t. \quad (1)$$

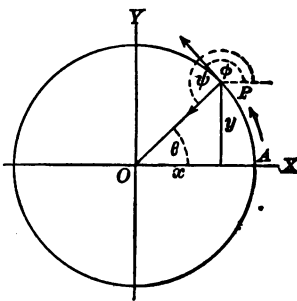


FIG. 45.

Differentiating

$$V_x = \frac{dx}{dt} = -a\tilde{\omega} \sin \tilde{\omega}t \quad \text{and} \quad V_y = \frac{dy}{dt} = a\tilde{\omega} \cos \tilde{\omega}t. \quad (2)$$

Differentiating again

$$a_x = \frac{d^2x}{dt^2} = -a\tilde{\omega}^2 \cos \tilde{\omega}t \quad \text{and} \quad a_y = \frac{d^2y}{dt^2} = -a\tilde{\omega}^2 \sin \tilde{\omega}t. \quad (3)$$

From equation (2) we get the co-ordinate velocities after any time t from the initial position; and from equation (3) we get the co-ordinate accelerations. The square root of the sum of the squares giving the total in both cases; thus,

$$v = \sqrt{V_x^2 + V_y^2} = a\tilde{\omega}, \quad (4)$$

and

$$a = \sqrt{a_x^2 + a_y^2} = a\tilde{\omega}^2. \quad (5)$$

The angle between the axis of X and the total velocity, or total acceleration, at any position is found from the tangent being equal to quotient of the Y component divided by the X component; or,

$$\tan \phi = -\cot \tilde{\omega}t, \quad (6)$$

and

$$\tan \psi = \frac{-a\tilde{\omega}^2 \sin \tilde{\omega}t}{-a\tilde{\omega}^2 \cos \tilde{\omega}t} = \tan(180^\circ + \tilde{\omega}t). \quad (7)$$

From equations (4) and (5) we have the *magnitudes* and from equations (6) and (7) the *directions with reference to the X axis*, of the total velocity or acceleration at any instant. From (7) it will be observed that the direction of the total acceleration is at all times toward the center of the circle, ψ being $180^\circ + \tilde{\omega}t$.

Consider the co-ordinate velocities. These are the velocities of the *projections*, on the X and Y axes, of the point as it moves along the circumference. With these projections the motion is subject to a *varying* acceleration. Take that along the X axis, $a_x = -a\tilde{\omega} \cos \tilde{\omega}t$, and substitute values for $\tilde{\omega}t$. When $\tilde{\omega}t = 0$,

$a_x = -a\omega^2$; when $\omega t = \frac{\pi}{2}$, $a_x = 0$; when $\omega t = \pi$, $a_x = a\omega^2$; and when $\omega t = \frac{3\pi}{2}$, a_x is again zero, and then the motion is repeated. The condition is known as the "simple harmonic motion," and is very frequently met with in the study of motion.

77. Polar Co-ordinates.—It is sometimes of advantage to use polar co-ordinates; in which case it is convenient to employ an angular measure of the velocity; a measure similar to that used in the preceding problem. Referring to Fig. 46, ds is the space traversed in time dt , whence,

$$v = \frac{ds}{dt} = \sqrt{r^2 \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2}. \quad (1)$$

The parts under the radical in equation (1) indicate the two component velocities which are usually employed with polar co-ordinates.

The rate of change of the angle θ is called the *angular velocity*, and we shall designate it by $\tilde{\omega}$, hence

$$\tilde{\omega} = \frac{d\theta}{dt}. \quad (2)$$

The rate of change of the radius vector is called the *radial velocity*, and we shall designate it by V_r , hence

$$V_r = \frac{dr}{dt}. \quad (3)$$

It is evident that the linear component, which is normal to the radius vector, of the total velocity and which we shall call $V_{\tilde{\omega}}$, is

$$V_{\tilde{\omega}} = r\tilde{\omega} = r \frac{d\theta}{dt}. \quad (4)$$

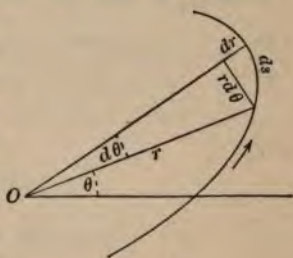


FIG. 46.

Thus we have two new component velocities, the square root of the sum of whose squares is equal to the total velocity as heretofore.

78. Polar Component Accelerations.—Referring to Fig. 47, the co-ordinates of any position P , of a point moving in a curved path, are

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta. \quad (1)$$

Differentiating equations (1), remembering that both r and θ are variables, we have for the component velocities

$$\frac{dx}{dt} = -r \sin \theta \frac{d\theta}{dt} + \cos \theta \frac{dr}{dt} \quad \text{and} \quad \frac{dy}{dt} = r \cos \theta \frac{d\theta}{dt} + \sin \theta \frac{dr}{dt}. \quad (2)$$

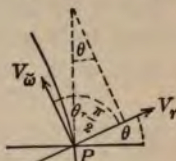


FIG. 47.

Differentiating equations (2) and reducing, we have for the component accelerations,

$$\frac{d^2x}{dt^2} = \left[\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \cos \theta - \left[2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right] \sin \theta. \quad (3)$$

$$\frac{d^2y}{dt^2} = \left[\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right] \sin \theta + \left[2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} \right] \cos \theta. \quad (4)$$

Equations (3) and (4) are the X and Y components of the total acceleration in terms used with polar co-ordinates.

Resolving these two components along the radius vector (angles marked in figure) and adding them algebraically, we have

$$\frac{d^2x}{dt^2} \cos \theta + \frac{d^2y}{dt^2} \sin \theta = \frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2. \quad (5)$$

Resolving them perpendicular to the radius vector, and adding, gives

$$\frac{d^2y}{dt^2} \cos \theta - \frac{d^2x}{dt^2} \sin \theta = 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} = \frac{1}{r} \cdot \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right). \quad (6)$$

Equation (5) is that component of the total acceleration which is directed *along the radius vector* and equation (6) is the one which is directed *along the normal* to the radius vector.

79. Rotary Motion.—When a point moves on the circumference of a circle the motion is called *rotary*. For rotary motion the methods of the preceding articles are much simplified; for, assuming the origin at the center of the path, the radius vector becomes a constant.

Under the above circumstances we have

$$\tilde{\omega} = \frac{d\theta}{dt}; \quad V_r = 0 \quad \text{and} \quad V_{\tilde{\omega}} = r\tilde{\omega} = r \frac{d\theta}{dt}. \quad (1)$$

$V_{\tilde{\omega}}$ has now the magnitude of the total velocity (the speed).

The accelerations, being the time-rates of the change of velocity, are found from (1). Calling the angular acceleration β ,

$$\beta = \frac{d\tilde{\omega}}{dt} = \frac{d^2\theta}{dt^2} \quad \text{and} \quad \frac{dV_{\tilde{\omega}}}{dt} = r \frac{d\tilde{\omega}}{dt} = r \frac{d^2\theta}{dt^2} = r\beta. \quad (2)$$

The radial acceleration has a value, as we found in Art. 74 (6), equal to $\frac{v^2}{r}$, and this acceleration is constantly directed toward the center of the circular path. Other forms are

$$\frac{v^2}{r} = \frac{r^2 \tilde{\omega}^2}{r} = r\tilde{\omega}^2. \quad (3)$$

With curvilinear motion the angles are measured in radians; thus, the angular velocities are in radians per second (written R/s) and the angular accelerations are in radians per second per second (written R/s²).

Problems VIII.

1. The path of a point is a circle. Compare the motion when the origin is taken at the center and when taken on the circumference.

2. An aeroplane is moving horizontally with a velocity of $3g$. A body let fall from it has this velocity and is accelerated by gravity. Find the body's speed and direction 4 seconds after being dropped.

3. The velocity of sound is V . Find the range when the projectile (angle of elevation θ) and the sound of discharge, reach a certain point, at the same instant.

4. A gun, angle of elevation 50° , has a range of 2449 ft. What was its muzzle velocity and the time of flight?

5. A hub of a wheel travels at a uniform speed. What points in the tire have the same speed?

6. Prove that when the angular velocity is constant, the acceleration along, and perpendicular to, the radius vector is independent of θ .

7. A point moves in a circle of radius 4 ft. with a uniform speed of 4 f/s. Prove that the acceleration is 4 f/s^2 .

8. What is the velocity and acceleration of a point on the earth's equator if the equatorial radius is 3962.8 miles?

9. Show that, under the acceleration of gravity, the time of descent of a point from the upper end of a vertical diameter of a circle, to any point on its circumference, along any chord, is constant, $= \sqrt{2D/g}$.

10. A projectile, fired from the top of a tower, at an angle of elevation of 45° , strikes the ground 60 ft. from the foot of the tower after 4 seconds. Find the height of the tower and the time before striking the ground if the projectile had been fired horizontally.

KINETICS.

CHAPTER IX.

THE LAWS OF MOTION.

80. Definitions. Units Used in Kinetics.—That part of Mechanics which treats of material bodies *in motion* under the action of the forces which produce the motion is called **kinetics**. In the study of kinetics we shall confine ourselves to the use of those units known as *gravitational* or *engineers'* units.*

These units are as follows:

The unit of *length, space, distance*, is the foot.

The unit of *time* is the second.

The unit of *force* is the pound; or, that force with which the earth attracts a standard quantity of matter called the pound mass.

The unit of *mass* is the quantity of matter possessed by anything which weighs g pounds; *i. e.*, the matter in any body for which the earth's attractive force is g pounds.

The unit of *velocity* is the foot per second; or, that motion which in one second would traverse a foot of space.

The unit of *acceleration* is the foot per second per second; or, that change of velocity which amounts in *one second* to one foot per second.

The unit of *work* is the foot-pound; or, that work which is done when a one pound mass is lifted through a space of one foot directly against the attractive force of gravity.

The unit of *power* is the horse-power (HP); or, 550 foot-pounds of work in *one second*, this being the average amount a horse can do in a second.

* Other units will be found defined in the Appendix. *p. 236*

The above definitions are tentative and will be expanded as we progress.

81. Laws of Motion.—The fundamental principles of Mechanics may be embodied in three statements known as Newton's Laws of Motion. These laws are based upon *observation* and *experiment* and were first presented, in materially their present form, by Sir Isaac Newton.

Law I.—*Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by force to change that state.*

Law II.—*Change of motion is proportional to the force applied, and takes place in the direction of the straight line in which the force acts.*

Law III.—*To every action there is always an equal and contrary reaction; or, the mutual actions of any two bodies are always equal and oppositely directed.*

82. The First Law.—The first law tells us that all matter is *inert*; that is, that in itself matter has no power of changing its condition of rest, or of motion in a straight line, and that an outside influence called *force* is necessary to cause motion in a body at rest, or to change the velocity of a moving body.

It is impossible to remove a body from the action of *all* force (for example, the force of gravity), so it is impossible to prove this law by experiment. The experiments we *can* try all tend to show that the law is true, and accepting it as fact we can derive from it two definitions, those of *force* and *inertia*.

Force is that which tends to change a body's state of rest, or of uniform motion in a straight line.

Inertia is that property possessed by matter by virtue of which *it continues* in its existing state of motion, or rest, unless acted on by force.

83. The Second Law.—From the second law we understand that the acceleration (change of motion per unit time) is directly

proportional to the force causing it and that it is directed along the line of action of that force. But we may go a little farther. These laws contemplate the effect of a force on a *single* body. Now it is readily proved by experiment that, the force which acting for one second causes a mass of *one* pound to move from rest, over a space of one foot, would cause a mass of *two* pounds to move from rest, over a space of only *half* a foot in one second. In both cases the space traversed *from rest* in the *first* second, is proportional to the magnitude of the acceleration produced. Therefore we may expand the second law a little and say in addition that, the accelerations caused by a force, F , separately applied to two different bodies, will be *inversely* proportional to the *masses* of the bodies. The second law may now be written; the acceleration varies *directly* as the force and *inversely* as the mass; or, a varies as $\frac{F}{m}$. Stated as an equation

$$a = k \cdot \frac{F}{m}, \quad (1)$$

a being the acceleration, F the force, m the mass, and k a constant.

We have all doubtless seen experiments tried which tend to prove the truth of this law.

Again, in accordance with this law, an infinitesimal force, acting on a *free*, stationary, body, no matter how huge, would cause the body to have a perceptible motion after a time. This seems curious because none of us have ever seen a body perfectly free from all force.

Again, the acceleration, which is caused by a force, is of the same magnitude, whether the body acted upon is at rest or in motion with any velocity whatever.

84. The Third Law.—Regarding the *third law*, we are all aware (from statics) of its truth in many cases; as, if a book rests on a table, it presses on the table with a force equal to its weight and the table presses upward on the book with an equal force. The

upward pressure of the table is called its resistance. The law is equally true of bodies in motion; as, the train resists the pull of the locomotive with a force equal to that the locomotive exerts on the train.

Another example is that of the attraction between the masses of two bodies. Take the familiar case of a body falling from a height to the ground. The cause of the fall is the attraction the mass of the earth exerts upon the body. This attraction causes the body to move toward the earth with an ever increasing velocity. The body attracts the earth with a force equal to that with which the earth attracts it; but the motion of the earth is insignificant, for, by the second law, the motion produced is inversely proportional to the masses, or the motion of the falling body is to the motion of the earth as the mass of the earth is to the mass of the falling body. A comparison of the masses will make clear why the earth *seems* to remain at rest. Of course for all practical purposes we assume the earth *does* remain at rest.

85. Mass.—With regard to the action of force upon them, material bodies differ only in the amount of matter they contain. When a large quantity of matter is confined in a small volume (a piece of lead), it is quite different, in appearance, from the same quantity of matter when it is occupying a greater volume (the same weight of air or feathers). The quantity of matter, the mass, is the same in the two cases, as the earth's attractive force, the weight, is the same; and, therefore, mass *varies directly with the density* of the material.

Again, the quantity of matter in *two* cubic inches of anything, as lead, is just twice that in *one* cubic inch; therefore, mass *varies directly with the volume*; and, designating the density by δ and the volume by V ,

$$m = \delta V. \quad (1)$$

Referring to Art. 83 (1), when a unit force acts for a unit of time on a unit mass, it imparts to the mass unit acceleration;

therefore, the value of k in that equation must be *one* with the units we have selected, and we may therefore write the equation

$$F = ma. \quad (2)$$

Now, when the earth attracts a free mass, it imparts to it an acceleration of g feet per second per second (see Art. 55). We know the earth attracts the standard pound mass with a *force of one pound* (the standard mass *weighs* one pound); therefore, the equation (from (2)), $W = mg$, must be true, and from it, $m = \frac{W}{g}$.

Substituting this value of m in (2), we get

$$F = \frac{W}{g} a. \quad (3)$$

We have now *derived* a value $\left(\frac{W}{g}\right)$ for mass and from it in gravitation or engineers' units, unit mass must weigh g pounds, for when *unit force* acts for *unit time* on *unit mass* it imparts *unit acceleration*.

It will be observed (assuming $g = 32 \text{ f/s}^2$) that the force, necessary to give an acceleration of one foot per second per second to a free pound, has a magnitude of half an ounce.

Equation (2) (or with the units we have chosen to use, equation (3)) is the most important equation in kinetics, as the *whole theory* is based on it.

Substituting $m = \frac{W}{g}$, in (1), we have $\frac{W}{g} = \delta V$, or $W = \delta g V$, from which we see that the weight varies directly with the density and the volume.

Al. 95
86. Momentum.—The quantity of motion a body possesses is called its momentum. If two bodies of equal mass are moving with the same velocity, side by side in a straight line, without rotation, the two together will have double the motion that one

of them has, or, momentum is proportional to the mass. If the velocity of a moving body be doubled it will have twice the motion it had at first, or, momentum is proportional to the velocity, hence:

The *momentum* of a moving body is the product of its mass and its velocity; and unit momentum is possessed by a unit mass moving at unit velocity (thus making the constant for the proportion equal to unity). Momentum is a vector quantity, as it is the product of a vector quantity (velocity) and a scalar quantity (mass).

87. The Rate, or Change, of Momentum.—The rate of *change of momentum* (or momentum acceleration) for a given time is equal to the force which acts during that time and causes the change; for,

$$\frac{d(mv)}{dt} = m \cdot \frac{dv}{dt} = ma = F.* \quad (1)$$

If we integrate the equation $F = m \cdot \frac{dv}{dt}$, we have, considering F to be constant, $\int F dt = \int m dv = mv - mv_0$

$$Ft \Big|_{t_0}^t = mv \Big|_{v_0}^v, \quad \text{or} \quad Ft \Big|_{t_0}^t = mv - mv_0, \quad (2)$$

or as the product of the force by the time it acts is called the *impulse*, the change of the momentum is equal to the impulse.

As F is constant during the time interval it must produce a continuous change of velocity. This being the case the change of momentum is also continuous, or it is a function of the time.

Many forces, such as that which causes a pitched ball to reverse direction and move off with an increased velocity when struck with a bat, can be measured only by the change of momentum they produce, the time during which the force acts being too short for measurement. Such forces are called *impulsive forces*.

* This is the complete mathematical statement of Newton's second law.

Problems IX.

1. If the unit of force is a pound and the unit of acceleration 1 f/s^2 , what must be the unit of mass?

~~2. What part of the force of one pound is a dyne? *~~

3. What force acting for 2 seconds on a free mass of 100 lbs. would give it a velocity of 50 f/s ?

4. What is the mass of a cu. ft. of sea water in gravitation units? In absolute units? *

5. What is the mass of a rod an inch square and a foot long, if the density varies directly as the distance from one end?

6. Through what space will a force of 10 lbs. move a free 50 lb. weight in 5 seconds, and what is the final velocity?

7. How long will it take an ounce force to move a pound mass $\frac{1}{8}$ ft. and what will be the final velocity? ²

8. The impulse given by a force in 2 seconds gives a free 100 lb. mass a velocity of $2g \text{ f/s}$. What is the force?

9. A 770 lb. force will give a free 60 ton mass a velocity of 50 m/h in how short a time?

10. A car weighing 64,000 lbs. and traveling 15 m/h is brought to rest in a space of 1210 ft. The retardation being uniform, find the retarding force.

11. A baseball weighing $5\frac{1}{2}$ oz. and moving at a velocity of 120 f/s moves off in an opposite direction at a velocity of 150 f/s when struck by the bat. Find the impulse of the blow. If the impact was over in $1/100 \text{ sec.}$, what was the force?

* See Appendix for definitions of these units.

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CHAPTER X.

WORK. POWER. ENERGY. IMPULSE. M. 75-101

88. Work.—In taking up kinetics, that part of Mechanics which treats of the motion of bodies while under the action of force, it will be convenient to understand the terms *work*, *power*, *energy* and *impulse*.

When a force causes a body to move, or when a body is moved against the action of force, work is done. The work done is proportional to the force which moves, or to the resistance to motion, and also to the whole distance through which the force has caused the body to move, or the resistance has acted; thus, the work done varies as the product of the force and the space. This proportion determines the unit of work; for, avoiding the constant implied by the proportion, we choose units such that when a *unit force* produces a *unit displacement* a *unit of work* has been done, and write the equation

$$\text{Work} = Fs, \quad (1)$$

where F is the force and s the total displacement caused by the force, or the total distance the body has been moved. The unit of force being the pound and the unit of space the foot, the name of the unit of work naturally becomes the foot-pound.

The *foot-pound* is the work done when a resistance of one pound is overcome through the space of one foot.

89. Work Due to a Constant Force.—The general equation for work is that of Art. 88, but it is true only when the displacement which takes place is in the line of action of the force. Should the motion be constrained, so that the point of application of the force moves in a direction making an angle with its line of action, the

work is +
or -
depending on
whether the
force and
displacement
are in the
same or
opposite
directions.

amount of work done would be due only to that component of the force whose line of action is in the direction of the constrained motion. For example, a point, constrained to move in a given direction, is acted on by a force F , which makes an angle θ with the direction of the allowable motion. The component of F along the line of the allowable motion is $F \cdot \cos \theta$, and s being the space traversed, the work done is $sF \cdot \cos \theta$ and should θ be equal to 90° , no work would be done. Mathematically no work can be done unless there is both motion and force, or no work is done when bodies are held in statical equilibrium. Thus a man who supports a heavy weight does no work unless he moves the weight.

Again, when motion is caused in a direction opposite to a force which opposes the motion, work is done. This work is done in overcoming the resistance R , the opposing force, and is equal to Rs units. *If pt of appn of force is displaced $\perp$ to force, it does no work*

It will be observed that *the kind of motion* does not affect the work done. In fact the work depends upon the *space* traversed by the point of application of a force (or its component) which is assumed to be of a constant magnitude and direction, and to act in the line of motion. (The force is usually a constant resistance.)

90. Work Due to a Variable Force.—If the point of application of a force, constant in magnitude and direction, moves along a curve, the component, which at any instant acts in the direction of the tangent to the curve, will be of a variable magnitude and will change in direction as does the tangent to the curve. Assuming the *direction* of a constant force to be parallel to the X axis (F_x in the figure) and that its point of application moves along the curve AB , the magnitude of its tangential component at any point along the curve will be equal to $F_x \frac{dx}{ds}$, the expression $\frac{dx}{ds}$ being the cosine of the angle the tangent, at any point, makes with the X axis. The work done while the point of application of this

*Forces which do no work: reactions of smooth curve on surface.
tension of light cord.*

component moves over a length ds of the curve, will be the product of the tangential component and the space ds ; or,

$$d(\text{work}) = F_x \cdot \frac{dx}{ds} \cdot ds = F_x \cdot dx,$$

$$\text{Work} = \int_{x_1}^{x_2} F_x \cdot dx. \quad (1)$$

Supposing the constant force to have been parallel to the axis of Y (F_y in the figure), the tangential component would have been $F_y \cdot \frac{dy}{ds}$, and the work would have been

$$\text{Work} = \int_{y_1}^{y_2} F_y \cdot dy. \quad (2)$$

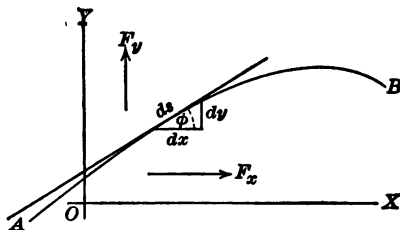


FIG. 48.

If we consider *both* forces to be acting, and the forces F_x and F_y , to be the X and Y components of any oblique force, the work done by that force will be equal to the algebraic sum of the works done by its two components, or the total work done by the oblique force whose point of application undergoes a displacement in a curve, is

$$\text{Work} = \int_{x_1}^{x_2} F_x \cdot dx + \int_{y_1}^{y_2} F_y \cdot dy. \quad (3)$$

For, if R is the resultant of two concurrent forces, P and Q , whose point of application undergoes a displacement s in any direction, and θ_1 , θ_2 and θ_3 are the angles which R , P and Q ,

respectively, make with the direction of the motion, then by Art. 17.

$$R \cdot \cos \theta_1 = P \cdot \cos \theta_2 + Q \cdot \cos \theta_3,$$

and as the work done is the product of these component forces and the space s , which is constant, the work done by the resultant of two (or any number of) forces is equal to the algebraic sum of the works done by the forces.

As in the preceding article this displacement is the space traversed by the point of application of the force.

91. Work Done in Moving Actual Masses.—The only work that is *necessary* to move a body through a space s is the product of the *resistance* to motion and the space through which the body is moved. As, for example, to move a body weighing ten pounds from the floor to a table three feet high, *requires* an amount of work equal to the weight (which is the resistance offered by gravity) multiplied by the height of the table; or, $10 \times 3 = 30$ foot-pounds of work. Again, to drag a body weighing W pounds a distance of s feet along a rough horizontal plane by a horizontal force, will *require* an amount of work equal to the product of the frictional resistance of the plane and the space; or, μWs foot-pounds.

The work, in each of the above cases, is the *least* possible consistent with the required transfer of the masses (see Art. 96).

92. Power.—The definition of power is the *rate of doing work*. The unit of power is called the horse-power (**HP**) and is equivalent to work at the rate of 33,000 foot-pounds per minute, or 550 foot-pounds per second.

For example, if a body of weight W is lifted h feet in t seconds, the work done is Wh foot-pounds, and as this took t seconds of time, the *rate* of doing the work was $\frac{Wh}{t}$ foot-pounds per second, or $\frac{Wh}{550t}$ horse-power. Now this result is the *average* value of the

rate of doing the work, for the weight was probably started slowly from rest, being accelerated during the first part of the journey and then retarded to rest during the last part, so that the spaces traversed during the different seconds were not the same. Therefore, if the power at any particular stage of the motion is required we must get it in a different way, as follows: find the velocity and acceleration (Chapter VII) at the instant for which we require the power; then in Art. 87 we find from the fundamental equation of motion,

$$F = ma = m \frac{dv}{dt} = \frac{d(mv)}{dt}, \quad (1)$$

Power = Fv
the force necessary to cause this acceleration in the mass involved. This force, *together with* the natural resistance to motion in the desired direction, will be the force which is acting at the instant. Also, if the mass should continue to move for one second with the velocity it had *at the instant*, it would move through a space equal to the magnitude of the velocity. Therefore, *at the instant*, work is being done at the rate of *force times velocity*. Or, in the above example, if the weight W has at any instant a velocity of v f/s and an acceleration a f/s², the force acting will be $(W + ma)$ and if the velocity remained constant, the space traversed in one second would be v feet, therefore the power at the instant is $(W + ma) v$ foot-pounds per second, or

$$HP = \frac{(W + ma)v}{550}.$$

It will be noted that the weight, W , was moved *vertically upward*, had the motion been horizontal W would not have appeared in the equation.

= capacity for doing work:
93. Energy.—When, for *any* reason, a body is capable of doing work, it is said to possess energy. Bodies possess energy for various reasons; as, due to their motion (momentum) they can, until it brings them to rest, do work against a resisting force; due to their position they may have stored energy which can be set free by a

change of position, as a weight at a height can do work if let fall; due to their temperature, as a hot bar of iron contracts when it cools and will exert considerable force at its ends if they are fixed; due to chemical composition, etc.

Energy is possessed by bodies whose *condition* is such that they can do work.

94. Kinetic Energy (K. E.).—The energy possessed by a body because of its motion is called kinetic energy. In accordance with Newton's first law, all bodies will continue in their existing state of rest, or motion in a straight line, unless acted upon by force. Referring to Chapter IX, a moving body, acted upon by an opposing force, is retarded; the retardation per second amounting to F/m . The body will finally be brought to rest; but not until it has traversed a space s , given by the formula $v^2 = 2as$ (Art. 55). The work done while the body is coming to rest, is equal to the resistance, $F = ma$, multiplied by the space, $s = v^2/2a$; or, the work done by a force F in bringing a body of mass m , which is traveling at a velocity v , to rest, is equal to $\frac{1}{2}mv^2$. Hence, a body moving at a velocity v is capable of doing work to the amount of $\frac{1}{2}mv^2$; or,

$$\text{K. E.} = \frac{1}{2}mv^2. \quad (1)$$

Again, from Art. 87, we have

$$F = m \cdot \frac{dv}{dt}, \quad \text{and} \quad v = \frac{ds}{dt}, \quad \text{or} \quad F = mv \cdot \frac{dv}{ds}.$$

Integrating

$$\text{Work} = \int_{s_1}^{s_2} F ds = \frac{m}{2} (v_1^2 - v_2^2) = \text{K. E.}, \quad (2)$$

work done = K.E. acquired

and from Art. 90, for motion in a curve,

$$\frac{m}{2} (v_1^2 - v_2^2) = \int_{x_1}^{x_2} F_x \cdot dx + \int_{y_1}^{y_2} F_y \cdot dy. \quad (3)$$

*PS = $\frac{m}{2}(v^2 - v_1^2)$
KE independent of s & t*

95. Work is Independent of the Time.—If a body is moved from rest through a space s and is then again at rest, the work

done is equal to Rs (R being the resistance); for, a body, subject to the resistance R , which is moved from rest by a force F , is accelerated in accordance with the formula, $F - R = ma$. If the body is to come to rest at the end of the path s , the force F must be removed at a point of the path, such that the resistance R , from there on, will be just sufficient to overcome the K. E. acquired by the body from the force F . Therefore, if the body, *while under the action of the force F* , moves a distance s_1 , the work done by F will be $(F - R)s_1 = ma_1s_1$. The velocity, at the point where F is removed, is given by the formula $v^2 = 2a_1s_1$; therefore, the body will, at that point, have K. E. equal to $\frac{1}{2}m \cdot 2a_1s_1$, which must be just sufficient to overcome R through the space, $s - s_1$, or $ma_1s_1 = R(s - s_1)$. These two values of the work being each equal to ma_1s_1 , may be equated, whence

$$Fs_1 - Rs_1 = Rs - Rs_1, \quad \text{or} \quad Fs_1 = Rs,$$

which shows, as F could do no work after being removed, that the whole work done is equal to Rs only; or, the *necessary* work is independent of the time. Of course the body could be hastened to its destination by allowing F to act longer and by making it larger, but this would require an additional retarding force, near the end of the path, to bring the body to rest at the proper place and this additional force would increase the total work done. This is frequently done in practice when speed is desirable, but even then if *sign* is given in accordance with the *direction* of the forces, the amount of work will equal Rs as before.

Static or positional energy
96. Potential Energy (P. E.).—The energy possessed by a body because of its position or configuration is called potential energy.

As examples, of bodies possessing potential energy, may be mentioned weights which, if allowed to fall, will do work; the spring of a wound watch which, if released, will do work; the water of a reservoir which, if allowed to flow out, will do work; compressed air, etc.

In all of the above examples, work has originally been done to put the bodies in such a condition as to possess potential energy, but having once been put in this condition, they are capable of restoring as much work, in returning to their original condition, as was necessary to put them into the condition of possessing the potential energy.

This is readily shown in the case of the falling body. From Art. 55, we have that to reach a height h , a body must be projected vertically *upward* with a velocity equal to $\sqrt{2gh}$, and also that a body, having fallen through a height h , acquires a velocity *downward* equal to $\sqrt{2gh}$. These velocities being equal, the kinetic energy, directed *upward*, possessed by the ascending body when it starts up, must equal that, directed *downward*, which is possessed by the falling body when it has fallen a distance h ; or, the work done in lifting a body of weight W to a height h is equal to Wh , and the kinetic energy of a body after falling through a height h is $\frac{1}{2}m \cdot 2gh = Wh$; therefore, the potential energy of a body at a height h is equal in magnitude to the necessary work done in placing it in that position. It will be observed that, if the velocities give sign to the work done, the sign of the potential energy will always be opposite to that of the work done to put the body in the possession of potential energy; also, that the *sum* of the K. E. and P. E. at any time, is a constant. (Principle of conservation of energy.)

97. Impulse.—By referring to Art. 87, we find that by integrating the equation, $F = ma = m \cdot \frac{dv}{dt}$, we get the time-integral of the force and it is equal to the change in momentum produced; or,

$$Ft \int_0^t = mv_1 - mv_2. \quad (1)$$

This equation is important when it is required to measure the impulse. Impulsive forces act for so short a time that they can be measured only by the change of momentum they produce.

Blow — impulse of force which acts on a body for some very small (and usually unknown) time and yet

For example, a ball of mass m moving at a velocity v , has momentum equal to mv , when it strikes normally a vertical wall. The first action is to compress both the wall and the ball until its motion stops; then, both the wall and the ball reassume their original shapes, and the ball springs back normally from the wall with a new velocity v_1 , which depends upon the value of the coefficient of restitution, or elasticity, of the materials. This coefficient which is determined experimentally and designated e , is a ratio between the velocity after impact and that before, and depends upon the material of the bodies in contact and *not* upon the *value of the* velocity. Its value will always be less than unity and greater than zero, as there is no substance which is either perfectly elastic or inelastic though many approach closely either limit.

We have then for the momentum after direct (normal) impact

$$mv_1 = emv, \quad (2)$$

and the whole impulse, given by the resistance of the wall, may be measured by the sum (v_1 being negative as it acts in an opposite direction to v) of the two momenta, or

$$Ft = mv + emv = mv(1 + e). \quad (3)$$

The time t is obviously very short, so F must be great in proportion and its value found from the above equation, if the time is known, will be the *mean* value of the force for the whole time of contact.

98. Oblique Impact.—Should the ball in the example of the preceding article strike the wall obliquely, at any angle θ with the *normal*, no change will occur in the component of the velocity which acts *along* the wall, *if both ball and wall are smooth*; the component of the velocity which is *normal* to the wall will be reduced, becoming e times what it was before impact. This makes

the angles between the lines of motion and the normal, before and after impact, unequal. They are related by the equation

$$\tan \theta_1 = \frac{1}{e} \cdot \tan \theta. \quad (1)$$

θ_1 being the angle between the normal and the line of motion *after* impact.

99. Problems.—(a) A spring is compressed from 20 to 17 inches by a weight of 471 pounds. How much work must be done to compress it another inch?

By Hooke's law the extension or compression of any material, within its elastic limit, is directly proportional to the extending (or compressing) force; therefore, $F=kx$, the k representing a constant and x the compression, and the element of work, Fdx , becomes $kxdx$, or

$$471 = k \cdot \frac{1}{2} \therefore k = 942, \text{ and work} = \int_{\frac{1}{2}}^{\frac{3}{2}} 942 x dx = 942 \frac{x^2}{2} \bigg|_{\frac{1}{2}}^{\frac{3}{2}} = 461\frac{5}{8} \text{ ft. lbs.}$$

(b) A locomotive developing 150HP and weighing 100 tons, is climbing a grade of 1 in 96, find the acceleration at the instant the speed is 15 m/h, if the frictional resistance is 14 pounds per ton.

At the instant, the locomotive is doing work at the rate of 150×550 foot-pounds per second; this work is done (1) against the frictional resistance, which is equal to 100×14 pounds, (2) against the component of the weight which acts down the grade and

which is equal to $100 \times 2240 \times \frac{1}{96}$ and (3) against the force,

$F=ma$, necessary to produce an acceleration a in the whole mass. As 15 m/h is 22 f/s, these three forces, all of which are opposed

to the motion, at the instant, are acting at the rate of 22 feet of space per second; therefore,

$$150 \times 550 = 22 \left\{ 100 \times 14 + \frac{100 \times 2240}{96} + \frac{100 \times 2240}{g (=32)} \cdot a \right\},$$

from which $a = \frac{31}{308} \text{ f/s}^2$.

(c) If a forging hammer weighs 2 tons and its blow extends over .01 second, what is the average force of a blow from a 6-foot fall?

Here $v = \sqrt{2g \cdot 6} = 8\sqrt{6} \text{ f/s}$ and $Ft = (mv - mv_0)$ where mv_0 is equal to zero, hence

$$F = \frac{2 \times 2240}{32 \times .01} \times 8\sqrt{6}, \text{ or } F = 112000\sqrt{6} \text{ lbs. or } 50\sqrt{6} \text{ tons.}$$

The data should *always* be used in our units, the pound, the foot, the second, etc., as in many cases the correct solution *cannot* otherwise be obtained with the formulæ given.

Problems X.

1. How much work is necessary to dig a well 28 ft. deep and 5 ft. in diameter if the earth weighs 150 lbs. per cubic foot?
2. A weight of 500 lbs. hangs by a chain 500 ft. long and weighing 2 lbs. per foot. How much work is necessary to lift the weight by winding up the chain on a windlass?
3. What must be the horse-power of a 10-ton automobile to be able to start from rest at the bottom of a hill a mile long, slope 1 in 15, and reach the top at a speed of 20 m/h, if the mean resistance is 12 lbs. a ton?
4. What horse-power is necessary to keep a 100-ton train moving at 30 m/h on a level track, the frictional resistance being 20 lbs. per ton?
5. If the power is shut off at the top of the hill, how far will

the automobile of problem 3 run on the level road beyond? (Resistance same as in problem 3.)

6. If the automobile of problem 3 is allowed to run freely down the hill against the resistance, with what velocity will it reach the bottom?

7. What potential energy did the automobile of problem 3 possess when at the top of the hill?

8. If a billiard ball is shot (without twist) from the middle point of the end cushion to the middle point of the side cushion, where will it strike the end cushion after "rounding the table"? $e = \frac{2}{3}$. Table $2a$ long and a wide. Omit friction and air resistance.

9. An ivory billiard ball is dropped from a height of 1 ft. on a smooth granite floor, how high will it go on the 3d rebound? $e = 0.8$. $e^{2/3} \approx 0.51$

10. A street car which weighs 20 tons runs at 15 m/h down a hill of 1 in 16 without acceleration. What power must it have to run up the hill at the same speed?

Handwritten: $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n x_i$

CHAPTER XI.

KINETIC EQUILIBRIUM.

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100. Definition and Explanation.—A moving body is said to be in kinetic equilibrium when its *state of motion* is unchanging. Also the forces which cause this condition must be in equilibrium.

A body's "state of motion" is obviously unchanging when it is moving, without rotation, at a uniform speed in a straight line, but the phrase "state of motion" must not be confused with the word "velocity," for a body moving at a constant speed in a circle, moves with an unchanging motion though with an accelerated velocity, Art. 71, as does also one which moves with a *uniform* acceleration in a straight line; also, either of these bodies though *rotating* with a constant angular velocity or a *uniformly* accelerated angular velocity, would still be in an unchanging state of motion.

The words "state of motion" then are to be understood as meaning the sort of motion, whether uniformly accelerated or not, that a body has *during the time selected to consider it*. We may, therefore, consider a body which is moving with a *variable* acceleration, to have an unchanging motion for an infinitesimal time interval and to be *for that instant* in kinetic equilibrium.

101. Inertia.—Referring to Chapter IX, by Newton's First Law of Motion, all bodies possess the property of resisting a change, either of their state of rest or of uniform motion in a straight line. This property is called inertia and it is a *force* which, like friction, lies dormant until an effort is made to change a body's state of rest or motion, when, again like friction, it acts *to oppose* the effort, or, *inertia is a force which acts during acceleration and in the direction opposite to the acceleration*.

p. 244

102. Measure of Inertia.—The Second Law of Motion provides us with the formula, $F=ma$, and by that law this formula means that, if a body of mass m , free from the action of all force, should be acted upon by a force F , it would be given an acceleration a during each second while the force acted, and also that from the instant the force was removed the body would continue on with a *uniform* velocity equal to the velocity it had at the instant the force was removed. As it is impossible to remove a body from the action of all force, let us assume a body to be suspended by a cord of infinite length, so that it may move freely in the horizontal plane. (The weight and tension of the suspending cord, being normal to this plane, can have no effect on the motion of the body in the plane.)

When in this condition, any horizontal force, however small, which may be applied to the body, will cause motion, which, as the motion starts from rest, means an acceleration in accordance with the above formula; or, *the least imaginable force will overcome the inertia of the body and give it an acceleration.* The force is measured by the product of the mass of the body and the acceleration it produces, but, by Newton's Third Law, for every action there is an equal and opposite reaction, therefore, the resistance to change of motion, or the inertia of a body,* must also be measured by the product of the mass and the acceleration; also from this law the inertia must act in a direction *opposite* to the direction of the acceleration. We have, therefore, the equation

$$\text{Force} = ma = -\text{Inertia.} \quad (1)$$

The negative sign being introduced on account of the *direction* of the inertia.

103. Kinetic Equilibrium.—From the preceding article it is clear that the single force causing acceleration in any body is in

* Calling inertia a *force* has so many advantages that this has been done throughout this book.

equilibrium with the inertia of the body and the body is said to be in kinetic equilibrium; but, also, the acceleration of any body acted upon *by a number* of forces is the same as though it were acted upon only by the resultant of those forces and therefore the body is in kinetic equilibrium, the *resultant* being in equilibrium with the inertia of the body. Again it will be observed that inertia acts only during *accelerated* motion; therefore, when a number of forces are acting upon a body which is moving with a *uniform* velocity, *those forces* must be in equilibrium and the body in kinetic equilibrium.

The motion we have been considering has been that of simple translation; a motion of translation being such that the paths of each particular particle, or molecule, of a body are equal and parallel. The force which causes such a motion must act through the center of gravity of the body, as must also any force, such as the body's inertia, which resists the motion. Were this not true there would be a couple which would cause rotation. Therefore for a simple motion of translation we may consider the center of inertia to be at the center of gravity of the body, that is, we may regard the body as a particle. This is true whether the body moves in a curved or a rectilinear path, as may readily be seen; but when bodies move in curved paths, they must be accelerated and the acceleration will act in some other direction than that in which the body, at any instant, is moving. Also the inertia will act in the opposite direction to the acceleration, or in a direction making an angle of 180° with the direction of the acceleration, Art. 101. But all the forces acting, including the inertia of the body, must form a system in equilibrium just as they do in rectilinear motion.

104. Resolution of Inertia.—It is frequently convenient to resolve the inertia force. For example the inertia of a body, moving at an accelerated speed in a curved path, may be resolved along and perpendicular to the tangent to the curve, in which case the component along the tangent will balance the force producing

the acceleration of the speed, while that perpendicular to the tangent balances the force producing the change of direction of the motion.

To illustrate, let a particle of mass m move at an accelerated speed in a circle of radius a . Referring to Art. 76, we see that the acceleration for a point moving at a uniform speed in a circle is $a\bar{\omega}^2$, where $\bar{\omega}$ is the angular velocity. At any instant, then, the force which causes a *change of direction* will be (equation (1), Art. 102) equal to $m \cdot a\bar{\omega}^2$ ($\bar{\omega}$ being the angular velocity at the instant), and this force will be directed toward the center of the circle, as in Art. 76. The force which causes the acceleration of the *speed* will, at any instant, act tangentially to the circle, and if the acceleration to the speed is equal to a , the force causing it will be equal to $m \cdot a$. The inertia components in each case will be equal to the tangential and normal components of the accelerating force and will act in the opposite directions to them (see Fig. 49). The total acceleration will act along the line shown (marked "total") and will be the resultant, equal to

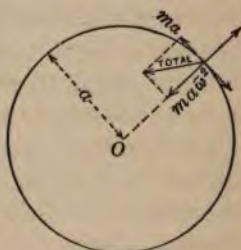


FIG. 49.

$$m\sqrt{(a\bar{\omega}^2)^2 + a^2},$$

of the two accelerations.

105. Inertia of Rotation.—When a rigid body rotates with a uniform angular velocity about a fixed axis, each particle of it is moving at a uniform speed in a circle. Each particle is undergoing acceleration *toward* the axis of rotation (Art. 104), and its inertia is acting *radially away* from the axis. The particle is in kinetic equilibrium in accordance with Art. 103. If the body rotates with an *accelerated* angular velocity, each particle undergoes a total acceleration as in Art. 104, the normal and tangential

components of which are (1) the normal, toward the axis and balanced by its opposite inertia as above, and (2) the acceleration of the speed, which acts tangentially and is balanced by an opposite tangential inertia.

Therefore, if we consider *each particle* of a rotating body *separately*, the conditions governing the motion of translation will hold true, but it will be noticed that the paths of the particles are of greater length as the particles are farther away from the axis of rotation and therefore the speed of the more distant particles must be greater, as all the particles must complete their circumferential path in the same time. Also the acceleration of the more distant particles must be greater and require a force of greater magnitude to cause it, when the body is rotating at an accelerated angular velocity. It would be perfectly correct to find the force and inertia for each separate particle, and add the results, to get the force or inertia, for the whole body, but it is simpler to make use of the mean values, to obtain which the calculus furnishes convenient methods; *i. e.*, suppose a body rotates with a constant angular acceleration, β , about any fixed axis. Consider any particle of mass, dm , at a distance, r , from the axis of rotation. The *linear* acceleration of the particle, at any instant, is $r\beta$, and the force causing it is $r\beta \cdot dm$. The moment of this force about the axis of rotation is $r^2\beta \cdot dm$, and the integral of this expression between the proper limits will be the sum of the moments, about the axis of rotation, of the forces acting on all the particles. Also, as the force in each case is equal to the inertia, the integration gives us the moment of the inertia of the whole body about the axis of rotation, or

$$\left. \begin{array}{l} \text{"rotational mass"} \\ \text{The moment of} \\ \text{the inertia} \end{array} \right\} = \beta \int_{\text{limit}}^{\text{limit}} r^2 \cdot dm. \quad (1)$$

The angular acceleration, β , being a constant.

It will be observed that the factor, $\int r^2 \cdot dm$ (usually indicated by I) is the same as was employed in the Calculus for the moment

Mon of in. = Σ mon. in. constituent parts

of inertia of any mass with regard to its axis of rotation. The factor, $\int r^2 \cdot dm$, or I , is *not* the moment of the *force* of inertia of a rotating body, but its *product*, with the angular acceleration, is *that moment*. The custom of calling I the moment of inertia has become general, but calculations involving the moment of the *force* of inertia, *always* require it to be multiplied by the angular acceleration. We need not consider the component of the radial inertia for it obviously has no moment about the center of motion as it passes through that point.

106. Radius of Gyration.—Having found the value of I by integrating the factor $\int r^2 \cdot dm$, of the moment of the inertia of a body, we have

$$I = \int r^2 \cdot dm = k^2 \cdot m,$$

and if we divide this value of I by the mass, m , of the body, we will have k^2 for the quotient. k^2 is called the *squared* radius of gyration, and k is the distance from the axis of rotation, that a mass, equal to the whole mass of the rotating body, would have to be concentrated in order that it might have the same moment of inertia as the rotating body. In other words, if all the mass of the rotating body were uniformly distributed on the circumference of a circle whose center was at the axis of rotation and whose radius was equal to k , the moment of the inertia of the mass thus distributed would be the same in magnitude as the moment of the inertia of the rotating body. *I & k are scalar quantities, always*

107. Equilibrium of Rotating Bodies.—Having found the *moment* of the force of inertia for rotating bodies, we must, in forming the equation indicating the kinetic equilibrium of these bodies, employ the *moment* of the force causing the accelerated motion; or, the equation for the kinetic equilibrium of rotating bodies should read

$$\text{Moment of } F = \text{Moment of the inertia force} = \beta I \cdot \beta k^2 m. \quad (1)$$

In this equation β is the *angular* acceleration. If we consider all the mass as uniformly distributed at a distance k from the axis of rotation, the *linear*, or *tangential*, acceleration, a , will be the product of the angular acceleration and the radius k , or $a=k\beta$, and substituting in (1) this value of $k\beta$, we have

Moment of F = moment of ma ,

or if the force F causing the acceleration is acting at a distance b from the axis of rotation, the equation becomes

$$bF = kma. \quad (2)$$

After referring to Art. 54 and Art. 79, we may write equations (1) and (2)

$$bF = k^2 m \beta = k^2 m \frac{d^2 \theta}{dt^2}, \quad (1a)$$

and

$$bF = kma = km \frac{d^2 s}{dt^2}. \quad (2a)$$

It is earnestly recommended that the student acquire a complete knowledge of the principles laid down in this chapter as most of what follows is based upon them.

Problems XI.

1. A piece of lead weighing $\frac{1}{2}$ lb. is thrown vertically upward. What is its inertia on the way up and on the way down?

2. A body moving with a uniform acceleration describes 63 ft. during the 4th second. Find its inertia.

3. A body uniformly accelerated covers 72 ft. while its velocity increases from 16 f/s to 20 f/s. Find its inertia.

4. A body describing the parabola, $y^2 = 4ax$, has a constant velocity v in the direction of the X axis. Find its inertia.

5. A fly-wheel, radius 3 ft., weight W , making 120 revolutions per minute (R. P. M.) is brought to rest in one minute. If the retardation is uniform what is its tangential inertia?

6. A fly-wheel, radius 6 in., weight W , making 360 R. P. M. is brought to rest after making 36 complete revolutions. Find its tangential inertia.

7. A fly-wheel, radius 6 in., weight W , starting from rest, makes 5000 complete revolutions in 3 minutes. Find the tangential inertia and the total radial inertia.

8. After making 2000 revolutions, a fly-wheel, radius 2.4 in., weight W , is making 500 R. P. M. Find the radial and tangential inertia and the time.

9. Show that the moment of the resultant inertia in example 7 about the axis of rotation is equal to $mr\beta$.

10. Show that the moment of the total inertia in example 8, about the axis of rotation, is equal to $mr\beta$.

by tangential in. he means 111.49"
radial " " 111.49"

Ponthe rule

Mom. in. abt. an axis of symmetry

$$= m \times \frac{\text{sum of sqs of } \perp \text{ semi axes}}{3 + 5}$$

i.e. as body is a rectangle (includes a right
" a strip
an ellipse includes circle
an ellipsoid includes sq

CHAPTER XII.

KINETIC EQUILIBRIUM.

RECTILINEAR MOTION. CONSTANT FORCE.

108. Acceleration Caused by a Constant Force.—We have seen, Art. 103, that a number of forces, acting on a mass and causing it to be accelerated, may be replaced by their resultant without changing the state of motion of the mass; also, the same article shows that the component of the resultant, whose line of action lies in the line of motion of the mass, is responsible for the linear acceleration of the mass. Therefore, if a mass, moving in a straight line, is being acted upon by a number of forces, its acceleration will be equal to the quotient of that component of the resultant of the forces which lies in the line of motion of the mass, divided by the mass of the moving body. Or, if θ be the angle between the line of motion of the mass and the line of action of the resultant of the forces, $R \cos \theta = F$, where F is the component of the resultant which causes the acceleration of the mass, and by the preceding chapter,

$$F = ma. \quad (1)$$

Referring to Chapter VII, we find from (1), Art. 54, and (1), Art. 55, the following values of a , for linear motion,

$$a = \frac{d^2s}{dt^2} = \frac{dv}{dt} = \frac{v dv}{ds}. \quad (2)$$

Substituting these values of a in equation (1), we have

$$F = ma = m \frac{d^2s}{dt^2} = m \frac{dv}{dt} = m \frac{v dv}{ds}. \quad (3)$$

Equations (3) are *general* equations of motion in a straight line or curves, and when F is a constant, there are but two variables and the equations may be directly integrated.

109. Equilibrium in Rectilinear Motion.—A body moving in a rectilinear path under the action of force, is in kinetic equilibrium in accordance with the principles of the preceding chapter and the body's inertia at any time is balanced by the force causing the acceleration. As an example, a quantity of coal, of weight W , is hauled up from a mine by means of a rope. The coal starts from rest and is uniformly accelerated until it acquires a speed of v f/s, which is the limit of speed for the engine which winds up the rope. After attaining this velocity, the coal moves at the uniform speed of v f/s until near the top of the shaft, where the engine is slowed down and the coal is retarded by gravity until it comes to rest at the mouth of the shaft. This motion is divided into three stages, (1) that in which the speed is accelerated, (2) that in which the speed is uniform, and (3) that in which the speed is retarded.

During the first stage the accelerating force is the tension, T_1 , of the rope, acting upward, less the weight, W , of the coal which acts downward. The tension at this stage is obviously the greater as the coal *ascends* from rest. The accelerating force, $T_1 - W$, must be balanced by the inertia of the coal, which, by Art. 102, must be equal to ma , and we therefore have as the equation of the motion, or as the statement of the kinetic equilibrium of the coal during the first stage

$$T_1 - W = \frac{W}{g} \cdot a_1. \quad (1)$$

During the second stage the acceleration is zero (hence *no inertia*) and equation (1), Art. 108, becomes (T_2 being the tension during this stage)

$$T_2 - W = 0; \quad (2)$$

or, the tension of the rope during uniform speed is equal to the weight. Equation (2) can scarcely now be called the equation of the motion, though we know from it that the speed is uniform. It

is, however, a statement of the equilibrium as inertia cannot exist without acceleration (see Chapter XI). An equation involving the velocity is, momentum is equal to the product of mass and velocity, but the force of momentum cannot be *utilized* without retarding the mass, and when retardation occurs we may go back to equation (1), Art. 108, and simply change the sign of the acceleration. This gives the equation of motion for the third stage, when the coal is being retarded to rest.

$$T_3 - W = -\frac{W}{g} \cdot a_3. \quad (3)$$

This equation shows that the tension of the rope is less, by the amount of the inertia, than the weight of the coal.

Looking at this problem from a slightly different point of view; the coal rests in a bucket while being hoisted from the mine, and the pressure of the coal on the bottom of the bucket must at all times be equal to the upward resistance offered by the bucket. During the first stage, the upward resistance, R_1 , of the bucket must equal the downward pressure of the coal, but this downward pressure is now equal to the weight of the coal *plus* its inertia, which also acts downward, the change of motion, or acceleration, being upward. Hence the upward resistance of the bucket will be

$$R_1 = W + \frac{W}{g} \cdot a_1, \quad (4)$$

which is the same in amount as the value of T_1 in (1) above.

It is readily seen that R will have the same value as T for the other stages of the motion. Again, if the coal rested on scales while being hoisted, its weight *by the scales* would equal T or R for the three different stages of motion.

110. Interacting Bodies.—When two or more independent masses are connected so that a natural linear motion of one will cause a linear motion of the other equal in amount to the first, the bodies are said to be *interacting*, and the equation of motion

is applicable to either or both of the bodies. For example, suppose two weights, W_1 and W_2 are connected by a smooth, inextensible cord which passes over a smooth pulley (see Fig. 50). Regarding first the greater weight, W_2 . It is in equilibrium under the weight, W_2 , acting downward, and, the tension, T , of the cord and the inertia of W_2 , which both act upward. Its equation of motion is

$$W_2 = T + \frac{W_2}{g} \cdot a; \quad (1)$$

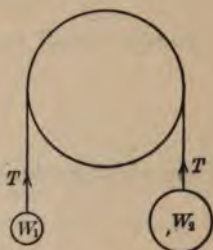


FIG. 50.

or, we may say the force acting is the weight minus the tension which is equal to the inertia, which also gives (1), the force, $W_2 - T$, acting downward as the motion will be caused in the direction of the greater weight.

Regarding next the lesser weight W_1 , remembering that its motion will be upward and that therefore its inertia will be directed downward; also, that the motion being upward, T must be greater than W_1 , we may write the equation of its motion

$$T - W_1 = \frac{W_1}{g} \cdot a. \quad (2)$$

It will be noticed that the values of a in equations (1) and (2) must be equal, for the two bodies, being connected by an inextensible cord must move at the same speed, though in opposite directions. Also, it is obvious that the tension of the flexible cord must be the same throughout (smooth pulley), and it should be obvious that it must act in opposite directions on either side of any point of it. We have therefore two simultaneous equations, containing but two unknown quantities, by solving which we get

$$a = \frac{W_2 - W_1}{W_2 + W_1} \cdot g,$$

$$T = \frac{2W_1W_2}{W_2 + W_1}.$$

Regarding the connected bodies together, the whole mass moved will be $\frac{W_2 + W_1}{g}$, and the force causing the motion will be $W_2 - W_1$. The equation of the motion of the whole mass will be, therefore,

$$W_2 - W_1 = \frac{W_2 + W_1}{g} \cdot a, \quad (3)$$

from which we derive the value of a as above,

$$a = \frac{W_2 - W_1}{W_2 + W_1} \cdot g.$$

Let us suppose that instead of hanging freely, as in Fig. 50,

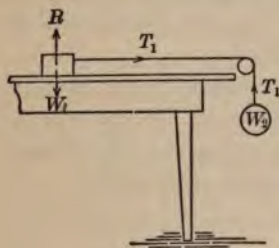


FIG. 51.

W_1 is resting on a smooth horizontal plane along which it is dragged by W_2 , which is hanging vertically from the cord which passes over a smooth pulley as shown in Fig. 51. The three equations may be formed as before, noting; that the weight W_1 acting vertically downward is now in equilibrium with the resistance, R , of the plane, which acts vertically upward; that both these forces are normal to the tension of the cord which causes W_1 to move, and that therefore they can have no effect, either to accelerate or (the table being smooth) to retard the system. Noting these differences the equations are

$$W_2 - T_1 = \frac{W_2}{g} \cdot a_1, \quad (4)$$

$$T_1 = \frac{W_1}{g} \cdot a_1, \quad (5)$$

$$W_2 = \frac{W_2 + W_1}{g} \cdot a_1, \quad (6)$$

from which we have the values

$$T_1 = \frac{W_2 W_1}{W_2 + W_1} \cdot$$

$$a_1 = \frac{W_2 g}{W_2 + W_1} = \frac{T_1 g}{W_1} \cdot$$

111. Work and Power.—In the example of Art. 109, the work done is obviously equal to (Chapter X) the product of the weight of the coal and the space through which it has been lifted; or, Wh . This may be shown by adding together the work done in each of the stages. *usefulness of graphs of wk*

Gravity acts continuously, therefore, whether in a condition of rest or of *uniform* rectilinear motion, an upward force, equal to the weight of any mass, is necessary to preserve its equilibrium, and under the action of any such force *not* equal to the weight, the mass will be accelerated. In the case of the coal we are considering, the necessary force during each stage is the tension of the rope. The value of this tension during the three stages will be given by equations (1), (2) and (3) of Art. 109. Now supposing the coal to have been accelerated during $\frac{2}{5}$ of its journey; to have traveled at a uniform speed during $\frac{2}{5}$ of it; and to have been retarded during the remaining $\frac{1}{5}$, so that it came to rest at the top of the shaft: then, as it was uniformly accelerated for twice as long as it was retarded, and as its velocity, during acceleration, changed from 0 f/s to v f/s, and during retardation, from v f/s to 0 f/s, we know that the retardation was of twice the magnitude of the acceleration, or that a_3 was equal to $2a_1$. The work done during each stage being equal to the tension of the rope multiplied by the space, we have

$$\text{1st stage Work}_1 = \left(W + \frac{W}{g} a_1 \right) \frac{2h}{5} \quad (1)$$

$$\text{2d " Work}_2 = W \frac{2h}{5} \quad (2)$$

$$\text{3d stage Work}_3 = \left(W - \frac{W}{g} \cdot 2a_1 \right) \frac{h}{5} \text{ (a retardation, negative) } \quad (3)$$

$$\text{or, the total work} = Wh \left\{ \frac{2}{5} + \frac{2a_1}{5g} + \frac{2}{5} + \frac{1}{5} - \frac{2a_1}{5g} \right\} = Wh. \quad (4)$$

Comparing equations (1) and (3) it will be observed that the work done in the 1st stage to overcome inertia was transformed into kinetic energy and possessed by the mass during the whole of the 2d stage, and then was given up during the retardation in the 3d stage, thus doing useful work; for, the inertia, during the 3d stage, acts through the space $\frac{h}{5}$ and in a direction opposite to that of the inertia in the 1st stage.

The *average* power necessary to lift W lbs., h ft., in t sec. would be $\frac{Wh}{t}$ ft. lbs. per sec.; or, $\frac{Wh}{550t}$ horse-power (IP); but while the coal was being accelerated the force necessary was T_1 of (1) Art. 109, or, work was being done to lift it *and also* to overcome its inertia. If the coal should travel for one second at the speed it had *at any instant* it would move over a space of v ft. (supposing v to be the velocity it had at the instant); therefore, at any instant during acceleration, work is being done at the rate of $v \left(W + \frac{Wa}{g} \right)$ ft. lbs. per sec., or

$$\text{IP} = \frac{Wv}{550} \left(1 + \frac{a}{g} \right). \quad (5)$$

Again, taking the first example of Art. 110, it will be observed that the system is continually accelerated by gravity and that it will continue to be so accelerated until acted upon by some *outside* force. The inertia of the weights is being overcome continuously, and the work done to overcome it is being stored up in the weights in the form of kinetic energy. In order to stop the system an amount of work, equal to the kinetic energy it possesses, must

be done, so that, when the system is brought to rest, we know that the work done in overcoming inertia and giving the system acceleration, has been balanced by the work done in overcoming the inertia when retarding the system to rest; the work in each case being equal to the kinetic energy the system possessed at the instant the retardation began. From this, and the preceding example, we see that, when a mass is moved *from rest* through a space s and is then *again at rest*, the inertia need not be considered in finding the work done, for the positive work done against inertia in accelerating the mass from rest is always balanced by the negative work done against inertia in retarding the mass back to rest; or, in other words, the quickness or slowness of the motion has nothing to do with the *necessary* work done in effecting the displacement; which means that, time does not enter into the work function.

In both of the above cases the motion has been *in the line of action of the resistance*; that is, it was vertical motion in the line of action of gravity. In every case the space should be measured along the line of action of the resistance and in the direction opposite to the direction of the resisting force; or, that component, of the resistance, which is along the line of motion must be used for the opposing force. For example (the space always being the *displacement* of the *center of gravity* of the mass), the work done in changing a heavy beam from a horizontal position on the ground to a vertical position with one end on the ground, is the product of the weight of the beam and the vertical distance its center of gravity has been moved; or, the beam being *uniform*, the work is half its length times its weight. *Frictional* resistances are *always* directly opposed to motion. The work done against gravity in putting a mass, which is at the bottom of a *smooth* hill, on the top of the hill, is the weight of the mass multiplied by the *vertical* distance to the top. A very long time might be taken to do this work by circling round and round the hill and getting a little higher on each turn; or, a quicker way would be to haul the mass directly

up the slope. The long way would require a lesser force than the short way, but that lesser force would have to be used through a longer distance to accomplish the required result. The total work done by either route would be the same in amount, for, the hill being smooth,* gravity is the only opposing force, and it acts through a space equal to the vertical height of the hill. This example should show clearly the distinction between *work* and *power*, which is the *rate* of doing work, the work here being a constant while the power necessary to do it may vary indefinitely, as the time taken to do the work may be of any length we choose.

112. Constrained Rectilinear Motion Produced by a Constant Force.—It is clear that when force acts on a mass the resulting line of motion will coincide with the line of action of the force, *unless the mass is prevented* from moving in that line. When it happens that the natural lines of motion are debarred for any reason, so that for a mass acted upon by a force the only allowable motion is along some line other than the line of action of the force, the motion is said to be *constrained*. Such is the motion of a body

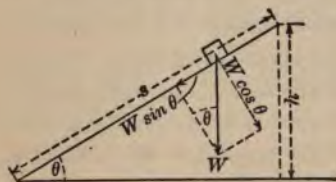


FIG. 52.

on a smooth inclined plane; for, the only force acting being gravity, the body will slide *down the plane*, though the line of action of gravity is *vertically* downward. Referring to Fig. 52, suppose the plane smooth and of inclination θ , also the weight of the body to be

W , which will act vertically downward, as shown. Resolving the weight W along and perpendicular to the plane, the component along the plane will cause the body to slide, while the perpendicular component, $W \cos \theta$, will hold the body pressed against

* The above conditions are obviously imaginary, as friction and other forces must always be considered when masses are moved.

the plane. All of which is in accordance with Art. 109, that that component of the resultant (in this case gravity alone) force which causes acceleration has its line of action coincident with the line of motion. It will be observed that, the plane being smooth, the normal component, $W \cos \theta$, does not affect the motion. Though had the plane been rough the product of this component and the coefficient of friction would give a retarding frictional force. The component *along* the plane, $W \sin \theta$, is a force causing acceleration and substituting its value in the equation of motion, $F = ma$, we have

$$W \sin \theta = \frac{W}{g} \cdot a,$$

which, as it should, for this is the component along the plane of the acceleration of gravity, gives us

$$a = g \sin \theta. \quad (1)$$

Using this value for a in the equations of Chapter VII, for velocity and space, assuming that the body starts from rest at the top of the plane

$$v = g \sin \theta \cdot t, \quad (2)$$

$$s = \frac{1}{2} g \sin \theta \cdot t^2; \quad (3)$$

also, eliminating t from (2) and (3); or, substituting in (1) Art. 55, gives

$$v^2 = 2g \sin \theta \cdot s. \quad (4)$$

From the figure, $s = h \operatorname{cosec} \theta$, whence the work done by gravity is

$$\text{Work} = W \sin \theta \cdot h \operatorname{cosec} \theta = Wh. \quad (5)$$

Again, we know the work done in overcoming inertia must be equal to the kinetic energy possessed by the mass when it reaches the

cases when overcoming inertia does not mean work

For rough plane $KE = W(\sin \phi \pm \mu \cos \phi)h$
 $= Wh \pm \text{friction work corres-}$
 $\text{ponding to dragging}$
 $\text{along horiz. projec-}$

bottom of the inclined plane; as, the body started from rest and was accelerated during the entire distance, we have, therefore,

$$\text{K. E.} = \frac{1}{2}mv^2 = \frac{W}{2g} \cdot 2g \sin \theta \cdot h \operatorname{cosec} \theta = Wh. \quad (6)$$

We see from equations (5) and (6) that the work done on the body in causing it to slide down the plane is *all* converted into kinetic energy, which the body possesses when it reaches the bottom. *Same applies to motion (under grav.) along curves.*

Had the plane been rough, the force acting *down* the plane would have been the difference between the component of gravity along the plane and the friction which would have acted along, but up, the plane. As stated above the amount of the friction would have been $\mu W \cos \theta$, so that the force and motion equation becomes,

$$W \sin \theta - \mu W \cos \theta = \frac{W}{g} a; \quad (7)$$

or, the acceleration down a rough inclined plane is

$$a = g(\sin \theta - \mu \cos \theta), \quad (8)$$

and, as before, by the equations of Chapter VII,

$$v = g(\sin \theta - \mu \cos \theta)t, \quad (9)$$

$$s = \frac{1}{2}g(\sin \theta - \mu \cos \theta)t^2, \quad (10)$$

$$v^2 = 2g(\sin \theta - \mu \cos \theta)s. \quad (11)$$

Had the body been projected *up* the plane, both the friction and the gravity component would have acted down the plane, and the equations would have become

$$v = V_0 - g(\sin \theta + \mu \cos \theta)t, \quad (12)$$

$$s = V_0 t - \frac{1}{2}g(\sin \theta + \mu \cos \theta)t^2, \quad (13)$$

$$v^2 = V_0^2 - 2g(\sin \theta + \mu \cos \theta)s. \quad (14)$$

113. The Wedge.—Both the wedge and screw are applications of the inclined plane. With the wedge, resistance is offered both

to the entrance of the wedge vertically, and to the horizontal separation of the fibers. Calling the total *normal* (the block resistance is similar to the effort of a pair of nut crackers) resistance R , as indicated in the figure, and remembering that the friction acts along the *inclined* surfaces, the work done against the resistances during a vertical entrance, dy , is

$$2R \sin \frac{\theta}{2} \cdot dy + 2\mu R \cos \frac{\theta}{2} dy.$$

This must be equal to the work done by the force F through the same space; or, Fdy . Hence

$$Fdy = 2Rdy \left(\sin \frac{\theta}{2} + \mu \cos \frac{\theta}{2} \right), \quad (1)$$

from which

$$R = \frac{F}{2 \left(\sin \frac{\theta}{2} + \mu \cos \frac{\theta}{2} \right)}. \quad (2)$$

It will be observed that if F is removed after the wedge is in, the vertical components of R will tend to force it out. The instant this tendency occurs, the friction reverses its direction, so putting F equal to zero and changing the sign of the friction in (1), we have

$$0 = 2R \left(\sin \frac{\theta}{2} - \mu \cos \frac{\theta}{2} \right), \quad (3)$$

from which

$$\mu = \tan \frac{\theta}{2}. \quad (4)$$

Equation (4) shows that unless $\tan \frac{\theta}{2}$ is less than μ the wedge will be squeezed out between the strokes of the sledge. ⁽²⁾

114. The Screw.—The screw does not properly come under the head of rectilinear motion yet it is a simple application of the



FIG. 53.

inclined plane. Taking the common screw jack with a square threaded screw, Fig. 54, let p be the pitch, so that one complete turn of the screw will advance it a distance p ; let r be the mean radius of the thread; F , the force employed to turn the screw, applied perpendicular to the turning lever at a distance a from the axis (see Fig. 54 (a)); and W the weight to be lifted. Then calling the horizontal turning effort at the extremity of the mean radius of the screw thread, H , we have by taking moments,

$$rH = aF, \text{ or } H = aF/r. \quad (1)$$

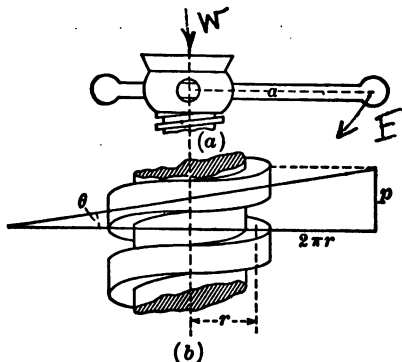


FIG. 54.

Now from equation (6) of Art. 39, noting that H acts at the angle $-\theta$, we have for the force (H in this case) necessary to drag a weight W up the incline,

$$H = \frac{aF}{r} = W \cdot \frac{\sin \theta + \mu \cos \theta}{\cos \theta - \mu \sin \theta}, \quad \text{or } Fa = rW \tan(\theta + \mu) \quad (2)$$

from which the ratio between F and W is

$$\frac{F}{W} = \frac{r(p + 2\mu\pi r)}{a(2\pi r - \mu p)} = \frac{d(p + \mu\pi d)}{2a(\pi d - \mu p)}. \quad (3)$$

d being the mean *diameter* of the screw thread.

In absence of friction force to act as pitch to circumference described by force

The efficiency of the screw is the ratio of the work done in raising the weight, to the work done by the force F ; or, if F turns through one circumference, the weight is raised a distance p , and the efficiency is $\frac{Wp}{2\pi aF}$, or from (2) and letting $\mu = \tan \alpha$

$$\text{Efficiency} = \frac{p(\cos \theta - \mu \sin \theta)}{2\pi r(\sin \theta + \mu \cos \theta)} = \frac{\tan \theta}{\tan(\theta + \alpha)}. \quad (4)$$

Problems XII.

1. A man who weighs 176 lbs. stands on a pair of scales in a lift. How much does he weigh by the scales if the elevator is being accelerated at the rate of 6 f/s²; if it is moving at a uniform speed of 20 f/s; if it is being retarded at the rate of 8 f/s²?

2. A bucket loaded with coal, the whole weighing 5 tons, is drawn up from a coal mine by a rope, in 30 seconds, from a depth of 1200 feet. The highest speed attained is 30 m/h and retardation begins the moment this speed is attained, the coal being at rest at the top of the shaft $4\frac{3}{4}$ seconds afterward. Find the greatest and least tension of the rope, the average horse-power required, and the horse-power just before the instant the speed was greatest.

3. An 11 lb. weight is attached by a cord which passes over a smooth pulley to two weights of 10 and 2 lbs., respectively. They are released from rest. Two seconds afterward the 2 lb. weight is detached. Find the acceleration and the velocity 2 seconds from rest, the retardation and the time till rest after the 2 lb. weight was detached, also the whole space traversed from rest to rest. What happens after the second rest?

4. Ten pounds rests on a rough horizontal table, μ equals $\frac{1}{2}$. It is attached by a cord passing over a smooth pulley to a 3 lb. weight hanging vertically and $3\frac{1}{2}$ feet from the floor. Find the acceleration and the tension of the cord, also how far will the 10 lb. weight move after the 3 lb. weight strikes the floor?

5. Suppose the table of problem 4 had been smooth.
6. A car which weighs 1680 lbs. runs down a slope of 1 in 30 at 12 m/h, the resistances at that speed being just sufficient to prevent acceleration. What horse-power must it exert to run *up* the slope at the same speed?
7. With what velocity must a 5 lb. weight be projected *up* a smooth incline of 1 in 15 and half a mile in length, to just reach the top?
8. The frictional resistances along an incline of 1 in 25 being 150 lbs. per ton, how much work is done when 2 tons are dragged up 100 ft. of it?
9. What force, directed toward the hinge, would be necessary to hold a smooth sphere, radius $\frac{1}{2}$ inch, in a pair of nut crackers? The sphere's center being 2 in. from the hinge and forces of 5 lbs., acting perpendicular to the arms, at a distance of 4 in. from the hinge.
10. The mean diameter of the thread of an ordinary square threaded screw-jack is 4 in., with a pitch of 1 in. What weight can it raise if 60 lbs. is applied to the lever arm at a distance of 4 ft. from the screw's axis? μ is equal to $1/10$.

Remarks on integration - 3 cases

CHAPTER XIII.

KINETIC EQUILIBRIUM.

RECTILINEAR MOTION. VARIABLE FORCE.

115. Force Varying as the Distance from a Fixed Point.—

When the effort of a force at a fixed point is to draw a mass directly toward the point, the force is *attractive*; for example, the force which draws a piece of steel to a magnet is attractive: when the effort is to cause the mass to move directly away, the force is *repulsive*; for example, the force between like magnetic poles is repulsive. The point from which the force acts is called the *center of force*. Suppose a mass is moved by an attractive force which varies directly as the distance of the mass from the center of force, and we wish to determine the motion. Referring to Fig. 55, let the mass, $W/g=m$, be at rest at a distance a from the

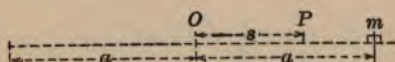


FIG. 55.

center of force, O , which we will take as the origin. Let the magnitude of the attraction for a *unit mass* at *unit distance* from the center of force be f (supposed known), and let s be the distance of the mass from the center of force at any time t from rest; then the force acting on the mass at P will be $-mfs$, which must, from the *general equation* (3) of Art. 108, be equal to $m \frac{d^2s}{dt^2}$, whence

$$\frac{d^2s}{dt^2} = -fs. \quad (1)$$

Multiplying (1) by $2ds$,

$$\frac{2ds \cdot d^2s}{dt^2} = -2fs \cdot ds. \quad (2)$$

Integrating

$$\left(\frac{ds}{dt}\right)^2 = -fs^2 \Big|_a^s = f(a^2 - s^2) = v^2. \quad (3)$$

Which gives the velocity at any distance s from the center of force. From (3) we see that the velocity is zero when s is equal to a and again when it is equal to $-a$; this shows that if the mass can pass through the center of force, it will continue on a distance a beyond, where the force, having there overcome the inertia acquired on the positive side of the origin, starts it back again. The mass will continue to oscillate, and will have its maximum velocity at the origin.

Extracting the square root of (3) and transposing, we have

$$-\frac{ds}{\sqrt{a^2 - s^2}} = f^{\frac{1}{2}} dt. \quad (4)$$

Choosing the negative sign because s decreases as t increases. Integrating (4), we have

$$\cos^{-1} \frac{s}{a} = f^{\frac{1}{2}} t + C. \quad (5)$$

The constant of integration being equal to zero, as s is equal to a when t equals zero, hence

$$f^{\frac{1}{2}} t = \left(\cos^{-1} \frac{s}{a} \right) \quad \text{or} \quad t = f^{-\frac{1}{2}} \cos^{-1} \frac{s}{a}. \quad (6)$$

From (6) the time from rest till the mass reaches the center of force is $\frac{\pi}{2f^{\frac{1}{2}}}$ or the time of complete ^{double} oscillation, from starting point back to the same point, is $\frac{2\pi}{f^{\frac{1}{2}}}$. It will be noted that this time is independent of the distance traversed or of the velocity, but de-

pendes *only* upon the value of f , the *absolute force or strength* of attraction.

From (6), we have

$$s = a \cos f^{\frac{1}{2}} t, \quad (7)$$

which differentiated gives

$$v = \frac{ds}{dt} = -af^{\frac{1}{2}} \sin f^{\frac{1}{2}} t. \quad (8)$$

From (7) and (8) we may find s and v in terms of t .

116. Harmonic Motion.—Simple harmonic motion may be defined as uniform circular motion resolved on a diameter of the circle. Referring to Art. 76, in which the acceleration of uniform circular motion was found by means of the co-ordinate accelerations, it will be observed that the values of the co-ordinate spaces traversed and co-ordinate velocities in that article, agree with equations (7) and (8) of Art. 115; also, noting that the time of a complete revolution of the point in Art. 76, agrees with the time of oscillation in Art. 115, it is apparent that force varying directly as the distance produces simple harmonic motion.

The following technical terms are employed in describing harmonic motion:

The *amplitude* of an harmonic motion is half the distance between two extreme positions; or, it is equal to the radius of the auxiliary circle.

The *period* is the interval of time between two successive passages through the same point in the same direction; or, it is the time of describing the auxiliary circle. *All three values, x , x' , x'' for $\omega(t+T) = \omega t + 2\pi$*

The *phase* is that part of the period which has elapsed since the mass was in its extreme positive position; or, it is the ratio of the time elapsed between the extreme positive position and any subsequent position, to the period. *or angle ωt*

Damped vibrations are those of a constantly diminishing amplitude. Such vibrations are those with which we are all familiar.

1. $y = a \sin(\omega t + \phi) = a \sin \omega t'$; (the $\omega t + \phi = \omega t'$ and t' is a variable
find amp + phase

for all actual motions are subject to frictional resistances which retard them. The equations of motion for damped vibrations are readily formed; for, supposing the motion of Art. 115 to be damped by a frictional force equal to μmg , which always acts in the opposite direction to the motion and must be applied accordingly and is therefore given the $\pm$ sign; we have by substituting in the general equation of motion

$$\frac{F}{m} = a = \frac{d^2s}{dt^2} = -fs \pm \mu g.$$

from which we derive the equation of space and velocity in the same manner as before.

117. Repulsive Force Varying as the Distance.—Had the force of the preceding article been one of repulsion and assuming that mass to be *at* the center of force when *t* is zero, the equation found by substituting in the general equation of motion, $F = m \frac{d^2s}{dt^2}$, Art. 108, would be

$$\frac{d^2s}{dt^2} = fs. \quad (1)$$

Performing the integration as before

$$\left(\frac{ds}{dt}\right)^2 = fs^2 + C. \quad (2)$$

The constant of integration of (2) will be the squared value of the velocity at the origin; calling this v_0 , (2) becomes

$$\left(\frac{ds}{dt}\right)^2 = fs^2 + v_0^2 = v^2. \quad (3)$$

Extracting the square root, transposing and integrating, we have

$$\frac{1}{f^{\frac{1}{2}}} \cdot \log \left(s + \sqrt{s^2 + \frac{v_0^2}{f}} \right) + C = t. \quad (4)$$

When t equals zero, s equals zero and $\therefore C$ is equal to $-\frac{1}{f^{\frac{1}{2}}} \cdot \log \frac{v_0}{f^{\frac{1}{2}}}$;
hence

$$\frac{1}{f^{\frac{1}{2}}} \left\{ \log \left(s + \sqrt{s^2 + \frac{v_0^2}{f}} \right) - \log \frac{v_0}{f^{\frac{1}{2}}} \right\} = t, \quad (5)$$

or

$$\frac{f^{\frac{1}{2}}}{v_0} \left(s + \sqrt{s^2 + \frac{v_0^2}{f}} \right) = e^{tf^{\frac{1}{2}}}, \quad (6)$$

from which

$$s = \frac{v_0}{2f^{\frac{1}{2}}} \left(e^{tf^{\frac{1}{2}}} - e^{-tf^{\frac{1}{2}}} \right). \quad (7)$$

Equation (3) shows that the velocity increases as s increases, while (7) shows that s increases as t increases, therefore the mass continues to get further and further away from the center of force as time goes on. The motion is *not* oscillatory.

118. Attractive Force Varying Inversely as the Square of the Distance.—When a body falls to the earth from a height so great that account must be taken of the variation in the force of gravity, due to the distance through which it acts, the varying force, which attracts the body toward the earth, is one which varies inversely as the *square* of the distance of the center of mass of the body from the center of mass of the earth. We will first take up the *general* case, considering only the positive space traversed. Referring to Fig. 55, let the body, of mass m , be *at rest* at a distance a from the center of force, O , which will be taken as the origin. Let the magnitude of the attractive force for a *unit mass* at *unit distance* be f , then, the force acting on the mass m , when at any distance s from the center of force, will be $-m \cdot \frac{f}{s^2}$. This force must be equal to the mass multiplied by the acceleration it possesses at the instant; or, by (3), Art. 108,

$$\frac{d^2s}{dt^2} = -\frac{f}{s^2}. \quad (1)$$

Multiplying by $2ds$, as in Art. 113, and integrating, we have

$$\left(\frac{ds}{dt}\right)^2 = v^2 = \frac{2f}{s} + C. \quad (2)$$

Noting that when $s=a$, the velocity is zero, we have $C = -\frac{2f}{a}$, or

$$\left(\frac{ds}{dt}\right)^2 = v^2 = 2f\left(\frac{1}{s} - \frac{1}{a}\right). \quad (3)$$

Equation (3) will give the velocity when the mass is at any *positive* distance s from the center of force.*

Extracting the square root and transposing (3) gives

$$-\frac{sds}{\sqrt{as-s^2}} = \sqrt{\frac{2f}{a}} \cdot dt. = \left[\frac{1}{2} \frac{a-2s}{\sqrt{as-s^2}} - \frac{a}{2} \frac{1}{\sqrt{as}} \right] dt \quad (4)$$

The negative sign is chosen because the space *decreases* as the time *increases*. Integrating (4), we have

$$\sqrt{as-s^2} - \frac{a}{2} \cdot \text{vers}^{-1} \frac{2s}{a} + C_1 = \sqrt{\frac{2f}{a}} \cdot t. \dagger \quad (5)$$

When t is zero, s is equal to a , or C_1 is equal to $\frac{\pi a}{2}$. Substituting this value of C_1 and solving for t , we have

$$t = \sqrt{\frac{a}{2f}} \left\{ \sqrt{as-s^2} - \frac{a}{2} \text{vers}^{-1} \frac{2s}{a} + \frac{\pi a}{2} \right\}. \dagger \quad (6)$$

Putting s equal to zero in (6) gives the time from rest till the body reaches the center of force, to be

$$t \Big|_{s=0} = \frac{\pi}{f^{\frac{1}{2}}} \left(\frac{a}{2} \right)^{\frac{3}{2}}. \quad (7)$$

* s *must* be positive for by equation (1) the acceleration will not change sign when s is negative, as we know it should.

† The number $-\frac{a}{2} \text{vers}^{-1} \frac{2s}{a}$ may be written $-a \sin^{-1} \sqrt{\frac{s}{a}}$; equivalent expressions.

Now it is obvious that if the mass passes through the center of force, the acceleration will become positive; but a negative value of s in equation (1) does not change the sign of the acceleration; therefore, *by equation (1)*, the acceleration *cannot* have a positive value. Again, by equation (3) the velocity is infinite when s is zero, and by equation (1) there is an infinite acceleration, indicating an infinite force. This shows the conditions to be physically impossible at the origin.* If the force is *repulsive*, the solution for v will show that the velocity never again becomes zero, but approaches the limit $\sqrt{\frac{2f}{a}}$ when $s = \infty$.

119. Velocity Acquired by Falling from a Great Height to the Earth.—Assuming that the center of force (the attraction of the earth's mass) is at the center of the earth, that the earth's radius is R , and that g is the attractive force for unit mass at the earth's surface, then, if a is the distance of the body from the earth's center when it starts to fall and s the distance at any time t , we have, $g = \frac{f}{R^2}$, or $f = gR^2$, hence the force acting on a unit mass at a distance s from the earth's center is $-g \frac{R^2}{s^2}$, which is equal to the acceleration of the unit mass, or

$$\frac{d^2s}{dt^2} = -g \frac{R^2}{s^2} \quad \times 2ds \quad (1)$$

Integrating as in Art. 117,

$$2 \left(\frac{ds}{dt} \right) d \left(\frac{ds}{dt} \right) = - \frac{2gR^2 ds}{s^2} \quad (2)$$

At the earth's surface $s = R$, hence

$$v^2 = 2gR \left(\frac{a-R}{a} \right). \quad (3)$$

Were a infinite we would have $v = \sqrt{2gR}$, or, almost 7 miles a second.

* It is suggested that the student attempt the complete solution of the problem.

120. Rectilinear Motion in a Resisting Medium.—Let the resistance vary as the square of the velocity; then, if the resistance offered by the medium to *unit mass* moving at *unit velocity* be f , and, if the force causing the motion would impart a uniform acceleration a to the mass if the motion were *not* delayed by the medium; then, the force acting on the mass at any time when its velocity is v , is $m(a - fv^2)$. Referring to the general equation (3), Art. 108, this force must equal $m \frac{dv}{dt}$, and the equation of the motion is

$$\frac{dv}{dt} = a - fv^2, \quad \text{or} \quad a \cdot dt = \frac{dv}{1 - \frac{f}{a} \cdot v^2}. \quad (1)$$

Integrating (1), we have

$$at = \frac{1}{2} \cdot \sqrt{\frac{a}{f}} \cdot \log \frac{1 + v \sqrt{\frac{f}{a}}}{1 - v \sqrt{\frac{f}{a}}} + C. \quad (2)$$

If the mass starts from rest, $v=0$ when $t=0$, hence $C=0$ and (2) becomes

$$e^{2t\sqrt{af}} = \frac{1 + v \sqrt{\frac{f}{a}}}{1 - v \sqrt{\frac{f}{a}}}, \quad (3)$$

from which

$$v = \sqrt{\frac{a}{f}} \cdot \frac{e^{t\sqrt{af}} - e^{-t\sqrt{af}}}{e^{t\sqrt{af}} + e^{-t\sqrt{af}}}. \quad (4)$$

Equation (4) gives the velocity in terms of the time. Using, for the equation of motion, another form (see (3), Art 108)

$$\frac{v \cdot dv}{ds} = a - fv^2, \quad (5)$$

and integrating as before we get

$$v^2 = \frac{a}{f} \left(1 - e^{-2fs} \right), \quad (6)$$

which gives the velocity in terms of the space traversed. Putting $v = \frac{ds}{dt}$ in (4) and integrating we get

$$s = \frac{1}{f} \cdot \log \left(\frac{\epsilon^{t\sqrt{af}} + \epsilon^{-t\sqrt{af}}}{2} \right), \quad (7)$$

which may be written

$$2\epsilon^{fs} = \epsilon^{t\sqrt{af}} + \epsilon^{-t\sqrt{af}}, \quad (8)$$

an equation giving the relation between the distance and the time.

From (4) it will be seen that as the time increases the velocity becomes more nearly uniform, becoming constant if t is infinite; or, by (1) the acceleration is zero when $v^2 = \frac{a}{f}$. From (8) we see that the space is also infinite when t is infinite.

If we put g for a in the above equations we will have the equation of the motion of a body falling through air close to the earth, for experiment shows that air resistance varies nearly as the square of the velocity (within speed limits). Of course, in any resisting medium, the shape of the moving mass may affect the flight; for example, a piece of tin and an iron sphere of the same weight would not travel the same path even though dropped from the same point. The above equations then are those for a heavy *particle*.

The student will note that in every case the formulæ have been derived from Newton's law, that the force is equal to the mass multiplied by its acceleration.

121. Energy of Vibration.—Referring to Art. 90, we see that the work done by a variable force is

$$\text{Work} = \int_{s_0}^s F ds, \quad (1)$$

and by (3), Art. 108, we find that $F = m \frac{v dv}{ds}$, substituting in (1)

we have

$$\text{Work} = \int_{s_0}^s F ds = m \int_{v_0}^v v dv, \quad (2)$$

which integrated gives

$$\text{Work} = \int_{s_0}^s F ds = \frac{1}{2} m (v^2 - v_0^2). \quad (3)$$

Equation (3) shows that the work done in any displacement is equal to the change of kinetic energy during the displacement.

As an example, the work done on the mass, moved as in Art. 115, noting that, as there the mass starts from rest, $v_0 = 0$, is

$$\text{Work} = \frac{1}{2} m f (a^2 - s^2) = \frac{1}{2} m v^2 = \text{Kinetic energy}. \quad (4)$$

The maximum kinetic energy is possessed by the mass when the space is equal to zero; the equation may be written in the form

$$\frac{1}{2} m f a^2 = \frac{1}{2} m v^2 + \frac{1}{2} m f s^2. \quad (5)$$

Equation (5) shows that *the sum of the kinetic and potential energies is at all times equal to a constant*, as it should be.

Problems XIII.

1. Find the space traversed in time t , by the projection on a diameter, of a point which moves at a constant speed V , in a circle of radius a , also time of one complete oscillation.

2. The stretch of an elastic string is directly proportional to the force stretching it. An elastic string, which is a foot long, stretches half an inch when a 2 lb. weight is supported by it; if the weight is pulled down till the string is $14\frac{1}{2}$ in. long, what is the acceleration at the instant the weight is released; what, when the string is just 13 in. long?

3. Find the maximum velocity of the weight in problem 2.

4. Find the time elapsed before the weight of problem 2 is back in its position of equilibrium.

5. Find the height to which the weight of problem 2 will ascend after the string has ceased to be taut.

6. A weight of 1 lb. rests on the middle of a smooth horizontal table. Two equal elastic strings, fast to opposite sides of the table, are each stretched 2 in. when attached to the weight, so that each string exerts a force of 4 lbs., in opposite directions, on the weight. The weight is now displaced a distance of 1 in. in the direction of one of the strings; find the maximum velocity of the weight and the time to make one complete oscillation.

7. The attraction of the mass of an homogeneous sphere for a particle *within* its surface varies directly as the particle's distance from the sphere's center. Suppose the earth homogeneous with a shaft straight through its center, how long would it take for a particle, released at its surface, to reach the earth's center? $\mu = \frac{g}{R}$

8. Find the velocity with which a body, falling from a great height h , would reach the earth's surface.

9. Derive, in terms of the space, the equation for the velocity of a heavy particle, ascending in the air against gravity, supposing it to start with a velocity v .

10. Find the greatest height to which the particle of problem 9 can reach.

CHAPTER XIV. Feb 94

KINETIC EQUILIBRIUM.

CURVILINEAR MOTION. Sl. 232 - 239

122. Curvilinear Motion.—When a point or mass has a motion in a plane curve, its position at any time is defined if, having the equation of its path, we know the law of the motion; that is, if the position of the radius vector is a known function of the time, with polar equations; or, if the abscissa and ordinate are known functions of the time with rectangular equations. The *direction* of the velocity at any instant is along the tangent to the path, and may, for the instant, be represented by a vector just as one would represent a rectilinear velocity.

Any curvilinear motion indicates an acceleration, and we have seen that in uniform circular motion the acceleration is directed each instant toward the center of the circle, its magnitude being equal to $a\omega^2$, or $\frac{v^2}{a}$, ω being the angular velocity, or, v the linear velocity; a being the radius of the circle. For *uniform* circular motion the acceleration is *all* radial, but it will be clear that any circular motion which is *not* uniform must have, in addition to the radial acceleration (which changes the *direction* of the motion only), a tangential acceleration which will cause a change in the speed of the mass. The total acceleration for a circular motion not uniform would be the resultant of these two component accelerations, and this resultant would *not* pass through the center of the circle. The same conditions hold for any curvilinear motion; that is, *at any instant* the radial component of the total acceleration is directed along the *radius of curvature* and passes

through the *instantaneous center of motion*, while the tangential component acts along the tangent to the path in the same or in the opposite direction to the motion as the speed is increasing or decreasing. These components of the total acceleration are called the *tangential* and *normal* components.

123. Components of Inertia.—At any instant the inertia of a mass, moving in a curvilinear path, acts in a direction opposite to the direction of the total acceleration and is equal in magnitude to the product of the mass and the acceleration (1), Art. 103.

The inertia, like any other force, may be resolved into components. It is convenient to resolve the inertia into components which, at any instant, are tangential and normal to the path of the mass, and these components will be equal in magnitude to the product of the corresponding component of the acceleration and the mass. The mass will be in kinetic equilibrium under the action of the component accelerating forces and the opposite component inertia forces.

Let the position of a mass in its path at any time be defined by the equations

$$x=f(t) \quad \text{and} \quad y=f_1(t). \quad (1)$$

The component velocities of the mass will be

$$\begin{aligned} v_x &= \frac{dx}{dt} = f'(t) = \frac{ds}{dt} \cdot \cos \phi \quad \text{and} \\ v_y &= \frac{dy}{dt} = f_1'(t) = \frac{ds}{dt} \cdot \sin \phi, \end{aligned} \quad (2)$$

where ϕ is the inclination of the motion to the X axis, and equals $\tan^{-1} \frac{dy}{dx}$.

The component accelerations at any time are,

$$\begin{aligned} a_x &= \frac{d^2x}{dt^2} = f''(t) = \frac{d^2s}{dt^2} \cdot \cos \psi \quad \text{and} \\ a_y &= \frac{d^2y}{dt^2} = f_1''(t) = \frac{d^2s}{dt^2} \cdot \sin \psi, \end{aligned} \quad (3)$$

where ψ is the inclination of the total acceleration to the X axis, and equals $\tan^{-1} \frac{d^2y}{dx^2}$.

These components are in each case parallel to the co-ordinate axes. The actual or total velocity at any time will be

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \quad (4)$$

and the actual or total acceleration,

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}. \quad (5)$$

The total velocity will act along the tangent to the path which is at the angle $\tan^{-1} \frac{dy}{dx}$ with the X axis, and the total acceleration will be along a line which makes the angle $\tan^{-1} \frac{d^2y}{dx^2}$ with the X axis, the tangential and normal components being

$$a_t = a \cos(\psi - \phi) \quad \text{and} \quad a_n = a \sin(\psi - \phi). \quad (6)$$

Therefore the component inertias are

$$ma_t = ma \cos(\psi - \phi) \quad \text{and} \quad ma_n = ma \sin(\psi - \phi); \quad (7)$$

or, by the method of Art. 74, the normal acceleration is $\frac{vd\phi}{dt}$; but the radius of curvature, $\rho = \frac{ds}{d\phi}$, or, $d\phi = \frac{ds}{\rho}$, substituting this value for $d\phi$, the normal acceleration, is $\frac{vds}{\rho dt}$. But $\frac{ds}{dt} = v$, hence the normal acceleration is $\frac{v^2}{\rho}$. This is a *general* expression and the normal component of inertia is thus $\frac{mv^2}{\rho}$.

124. Centrifugal and Centripetal Force.—The forces which always act normal to the motion when a mass moves in a curvilinear path have been given special names; *i. e.*, the normal com-

ponent of the *inertia*, which always acts *away* from the center of motion, is called *centrifugal* force, and the normal component of the *accelerating force*, which always acts *toward* the center of motion, is called the *centripetal* force. Each is *equal* to the other in *magnitude* but *opposite* in *direction*.

We find in Art. 76, that for a mass moving at a uniform speed in a circle the acceleration is constant and equal to $a\bar{\omega}^2$; or, $\frac{v^2}{a}$.

From Newton's law then we may write down the value of the centrifugal or centripetal force which acts on a mass m which moves at a uniform speed in a circle.

$$F = ma\bar{\omega}^2 = m \frac{v^2}{a}. \quad (1)$$

Equation (1) may be put in a number of different forms; as, $m = \frac{W}{g}$ and $\bar{\omega} = \frac{2\pi}{T}$, where T is the time of one complete revolution; also if n is the number of revolutions per second, $2\pi n = \bar{\omega}$, and we may write (1) in the following forms:

$$\left. \begin{array}{l} \text{Centrifugal} \\ \text{Centripetal} \end{array} \right\} \text{force} = \frac{Wa\bar{\omega}^2}{g} = \frac{Wv^2}{ga} = \frac{4\pi^2 aW}{gT^2} = \frac{4\pi^2 n^2 aW}{g}. \quad (2)$$

We all know that a small weight whirled in a circle at the end of a cord keeps the cord taut. That is the effect of the centrifugal and centripetal forces acting in the cord; for the former tends to pull the weight *away* from the center of motion, and the latter (the resistance offered by the person holding the cord) resists the effort, the consequence being that the weight moves in a circle. If either force fails to act the motion ceases to be circular. These conditions hold for *any* curvilinear motion, but motions other than uniform circular motion have *in addition* tangential acceleration which varies the speed.

125. Centrifugal Force of Earth's Rotation.—The centrifugal force due to the rotation of the earth on its axis, is one of the causes for the change in the value of g at different latitudes.

Another cause for this change is that the polar diameter of the earth is less than the equatorial diameter and the center of force of the attraction of the earth's mass is at the center of the earth.

Referring to Fig. 56, let the circle $ADBE$ represent a meridian section of the earth, let AB be the projection of the equator and DE the axis on which the earth rotates. A mass at P will then move in a circle of radius CP and make a complete revolution in a day. The angle POB is equal to the latitude of the point P , and

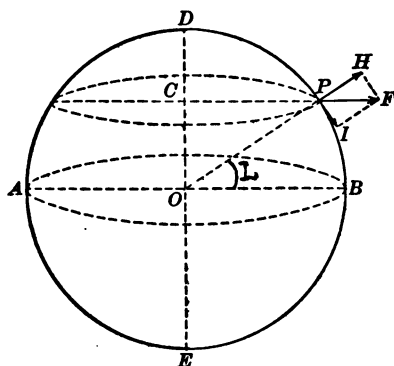


FIG. 56.

we will call this latitude L . From the figure, $CP = R \cos L$ (R being the radius of the earth).

Now if T is the number of seconds in a day, by (2), Art. 124, we have

$$\text{C. F. on mass } m \text{ at Lat. } L = m \frac{4\pi^2 PC}{T^2} = m \frac{4\pi^2 R \cos L}{T^2}. \quad (1)$$

The C. F. acting on a unit mass at the equator is $\frac{4\pi^2 R}{T^2}$, and calling this value f , we have from (1)

$$\text{C. F. on mass } m \text{ at Lat. } L = mf \cos L. \quad (2)$$

Equation (2) shows that the centrifugal force (at any place), which is due to the earth's rotation, varies directly as the cosine of the latitude of the place.

The centrifugal force acts in the direction, PF , as indicated in the figure, and may be resolved along and perpendicular to a tangent as shown. The resolved part $PH = mf \cos^2 L$, will act directly to reduce the weight of a mass at P , as gravity acts toward O along the radius PO . The resolved part $PI = mf \cos L \sin L$, will tend to pull the mass toward the equator. This latter force accounts for the spheroidal shape of the earth. It will be noted that the resolved part PH diminishes as the latitude increases which causes an increase in the value of g as the latitude increases.

126. Railway Curves. Banked Tracks.—The velocity in any curvilinear motion may become so great that the mechanical devices, by which the radial accelerating force is opposed, are not sufficiently strong to withstand the stress put upon them; for example, the cord which holds the whirling weight in Art. 124, may be broken because the tension it is under, due to the centrifugal and centripetal force, is greater than the resistance it is capable of offering; or, a railway train rushing round a curve at a great speed, will be likely to shove the track radially out from the center of the curve, or to topple over outward and thus cause disaster.

In the case of the railway train, the flanges of the wheels grind on the *outer* rail (rail farthest from the center of the curve) the flanges offering a centrifugal thrust *outward* and the rail resisting this with a centripetal thrust *inward*. These being the conditions when the rails are at the same level. In order that there may be no lateral thrust on the rails, the outer rail is raised to a higher level than the inner one. The resultant forces will then act as shown in Fig. 57. The outer rail should be raised enough to make the resultant of the C. F. and weight per-

FIG. 57.

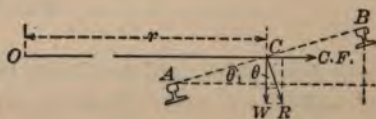


FIG. 57.

pendicular to a line between the two rails. The weight, W , of the train acts vertically downward and the C. F. acts horizontally and *away* from the center, O , of the curve. The resultant, R , of these two forces will act as shown, and will be in equilibrium with the track resistance which will act opposite in direction to R . The tangent of the angle R makes with the vertical will be

$$\tan \theta = \frac{\text{C. F.}}{W} = \frac{Wv^2}{grW} = \frac{v^2}{gr}. \quad (1)$$

Now if for a certain speed v , the pressure on the rails is to be exactly normal to a line AB between them, the angle BAD must be equal to the angle RCW , and knowing the distance AB , the "gauge," we may readily find the height, BD , that the bearing surface of the outer rail must be above the bearing surface of the inner rail.

The above is readily extended to banked tracks (racing tracks, etc.), the radius of the curve being the radius at any instant of the curve described by the moving body.

127. The Conical Pendulum.—A small mass, attached by a cord

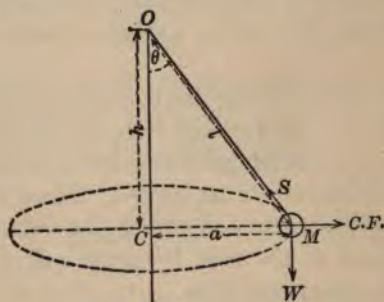


FIG. 58.

to a fixed point, and revolving in a horizontal circle, the center of which is directly below the fixed point to which the cord is attached, is called a conical pendulum; ideal conditions being, a

weightless, inextensible cord, and no air or frictional resistance. Fig. 58 represents such a pendulum, the length of the cord being l and the mass describing a horizontal circle of radius a . The mass is in equilibrium at any point of the circle under the action of the forces shown, the weight W , the C. F., and the tension, S , of the cord. The triangle OCM is a triangle of forces, and if T is the time in seconds of one complete revolution, we have

$$\frac{a}{h} = \frac{\text{C. F.}}{W} = \frac{4\pi^2 a}{gT^2} = \frac{Wa\tilde{\omega}^2}{Wg}, \quad (1)$$

from which

$$h = \frac{gT^2}{4\pi^2} = \frac{g}{\tilde{\omega}^2}. \quad (2)$$

From (2) we see that, for a given time of revolution, h is independent both of the weight of the mass and the radius of the circle, also, that h is inversely proportional to the square of the angular velocity. We have also from (2) that $\tilde{\omega} = \sqrt{\frac{g}{h}}$ or, that the time T of one complete revolution is

$$T = \frac{2\pi}{\tilde{\omega}} = 2\pi \sqrt{\frac{h}{g}}, \quad (3)$$

which also comes directly from (1). From the figure we have

$$\cos \theta = \frac{h}{l} = \frac{gT^2}{4\pi^2 l} = \frac{g}{\tilde{\omega}^2 l}, \quad (4)$$

and

$$S = W \sec \theta = \frac{4W\pi^2 l}{gT^2} = \frac{W\tilde{\omega}^2 l}{g}. \quad (5)$$

A practical application of the principles of the conical pendulum occurs in the "governor" of the steam engine.

128. Motion in a Vertical Circle.—If a small mass, attached by a cord to a fixed point, moves in a vertical circle, with such a velocity, when it passes the point A of Fig. 59, that the C. F.,

acting upward, *just* exceeds the weight acting downward, so that the mass, instead of falling when it reaches *A*, will continue on in the *circular* path, we would have

$$W = \frac{W}{g} r \tilde{\omega}^2 = \frac{W v^2}{g r}, \quad (1)$$

from which

$$\tilde{\omega}^2 = \frac{g}{r}, \quad \text{or} \quad v^2 = rg. \quad (2)$$

Therefore, in order that a mass may continue to move in a vertical circle, it must have, at the highest point of the circle a velocity which *just* exceeds the value given in (2). Now the velocity acquired by falling through a height *h* is $\sqrt{2gh}$, so the velocity given by (2) is equal to that acquired by falling

through a height equal to $\frac{r}{2}$.



FIG. 59.

When the mass has traversed a space equal to a semi-circumference from *A*, that is when it reaches *B*, its velocity has been increased *by falling through* the height $2r$; therefore, at *B*, the velocity must just exceed that acquired by falling through a

height $\frac{5r}{2}$, or the velocity at *B* just exceeds $\sqrt{5gr}$.

The above velocities *must* be reached before a mass *can* continue to move in a vertical circle as described.

When a mass has any velocity exceeding the above, we have by the principle of the conservation of energy, Art. 96, that the sum of the kinetic and potential energies at any instant is equal to a constant. Assuming the point *B* as the level of zero potential, a body at any point *P* of the circular path has potential energy equal to $W \cdot CB$, and if the velocity at *B*, where the potential energy is zero, is v_B , the sum of the two energies at that point,

or the constant, is equal to $\frac{1}{2}mv_B$, therefore, calling the velocity at P , v_P , we have

$$\frac{1}{2} \frac{W}{g} v_P^2 + W \cdot CB = \frac{1}{2} \frac{W}{g} v_B^2, \quad (3)$$

or

$$v_P^2 = v_B^2 - 2g(CB). \quad (4)$$

If the velocity at B is equal to that acquired by falling through a height H , equation (4) becomes

$$v_P^2 = 2g(H - CB). \quad (5)$$

Again, when the mass is at P the C. F. is $\frac{Wv_P^2}{gr}$ and acts radially, also a component of W , equal to $W \cdot \cos \theta$, acts radially, therefore the tension S of the cord, for that position of the mass, is

$$S = \frac{Wv_P^2}{gr} + W \cos \theta. \quad (6)$$

Problems XIV.

1. A 4 oz. weight, supported on a smooth table, is whirled in a horizontal circle at the end of a cord 5 ft. long. If the cord will just sustain a tension of 2 lbs. without breaking, how many revolutions per minute can the weight make without breaking the cord?

2. Find the accelerating force for a unit mass which is moving in a circle of 5 ft. radius at a uniform speed so that it completes a revolution in 5 seconds.

3. A weight of 10 lbs. rests on a rough, horizontal disc (coefficient of friction .4) which is caused to revolve about a vertical axis through its center with an angular velocity of 6. If the weight is just on the point of sliding, how far is it from the center of the disc?

4. Find the effect of the centrifugal force in reducing gravity

in latitude 60° , assuming the earth to be a sphere of radius 4000 miles.

5. Trains travel at 30 m/h round a curve of radius 1000 ft. How much must the outer rail be raised that there be no lateral resistance? The gauge being 4 ft. 8 in. and g equal to 32.

6. A circular bicycle track is $\frac{1}{15}$ mile long. At what angle should it be banked for a speed of 30 m/h? Neglect friction.

7. A 1 lb. weight is whirled in a horizontal circle at the end of a cord 2 yds. long, the cord being under a tension of 3 lbs. What is the time of a complete revolution?

8. Find the change in height of a conical pendulum which is making $\frac{4}{3}$ revolutions per second, if the speed is increased 2%.

9. A 4 oz. weight is whirled in a vertical circle at the end of a cord 3 ft. long, having just sufficient velocity to travel in the circle. What is the velocity (1) at the highest, (2) at the lowest point, and (3) when opposite the center of the circle?

10. A plumb-line hangs from the top of a car going round a curve of 500 ft. radius at 30 m/h. At what angle is it inclined to the vertical?

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CHAPTER XV.

KINETIC EQUILIBRIUM.

Sl. 264-271

CURVILINEAR MOTION. CONSTANT FIELD.

129. Curvilinear Motion.—In this chapter we will continue the consideration of the motion of a mass when its path is curvilinear. If a mass moves in a curved path, the force which causes the change of direction of the motion (the accelerating force), must, at all times, be perpendicular to, or oblique to, the path; for, by Newton's law a moving mass will continue its motion in a *straight line* unless acted upon by force; also, if the force acts *in* the line of motion the only effect it has, as we have seen, is to accelerate or retard the speed. If the force which acts on a moving mass is constant, both in magnitude and *direction*, and acts perpendicularly or obliquely to the line of motion of the mass, the direction of the motion will change continuously from the moment the force begins to act *until* the motion coincides with the line of action of the force, after which the motion will continue in the straight line which *is* the line of action of the force; also, from the instant the rectilinear motion begins, the force will only accelerate the speed of the moving mass. Therefore, if motion continues indefinitely to be curvilinear, the *direction* of the force which causes the change of direction of the motion must also change continuously as the mass moves through space. Thus curvilinear motion will continue indefinitely or not as the *direction* of the accelerating force is continually *changing* or not.

130. Fields of Force.—A place where force exists is a field of force. In the vicinity of a magnet any particle of iron is attracted and the space within the limits of the magnet's attractive

force is known as the *magnetic field*. From our point of view (*i. e.*, the *motion* of the particle of iron within the limits of the attraction) such a field would be called a **central field**, for the accelerating force, anywhere within the field's limits, would always act toward the center (magnetic pole). If a particle of iron should be projected *across* the lines of force in such a field it would continue to move in a curve, as we shall see later.

A field at all points of which the accelerating force is *uniform in amount* and *parallel in direction* is called a **constant field**. In a constant field the accelerating force acting on any mass is constant both in magnitude and direction, and a body projected into such a field, in a direction crossing the field's lines of force, would move in a curve until its line of motion was along the lines of force of the field, after which it would move in a straight line. An example of a constant field is the attraction of gravity at any place. Of course the lines of action of gravity are all directed toward the center of the earth, but at any place (within limits of area) they may be considered parallel for all practical purposes.

The motion of *falling* bodies, bodies lifted *vertically*, bodies *constrained* to move in straight paths under the action of a constant force (such as bodies sliding down an inclined plane, etc.) all come under motion in a constant field, but bodies *projected* into a constant field, along a line oblique to the lines of action of the force of the field, move in curves. Perhaps the most familiar example of such motion is that of a projectile (see Art. 75), when its motion is considered apart from the atmospheric resistance.

Consider a mass m to be projected with a velocity V_0 at an angle α with the lines of force in a constant field. If the field's force is gravity and α the angle of elevation, we have at any time the X co-ordinate of the accelerating force equal to zero and the Y co-ordinate equal to mg . The co-ordinate equations of motion are then

$$m \frac{d^2x}{dt^2} = 0 \quad \text{and} \quad m \frac{d^2y}{dt^2} = -mg. \quad (1)$$

Integrating and substituting the X and Y co-ordinates of the velocity at the origin for the constants of integration, we have

$$\frac{dx}{dt} = 0 + V_0 \cos \alpha \quad \text{and} \quad \frac{dy}{dt} = -gt + V_0 \sin \alpha. \quad (2)$$

Integrating again, both constants being equal to zero,

$$x = V_0 \cos \alpha \cdot t \quad \text{and} \quad y = -\frac{1}{2}gt^2 + V_0 \sin \alpha \cdot t. \quad (3)$$

Eliminating t , we have as in Art. 75, the equation of the trajectory in vacuo, with the origin "at the gun," is *

$$y = x \tan \alpha - \frac{gx^2}{2V_0^2 \cos^2 \alpha}. \quad (4)$$

131. Motion on a Smooth Vertical Circle Under Gravity.—Let a small ring of mass m be strung on a smooth wire bent into a circle of radius r , then, if the ring is moved around to any position B and released, Fig. 60, it will slide down the arc BO under the action of gravity, the conditions at each point being those for a body sliding down an inclined plane. When the ring reaches the lowest point O , it will have a velocity v , equal to that it would have acquired by falling freely through the vertical height DO , Art. 122, but which is (at O) horizontal in direction. At O the ring's kinetic energy is $\frac{1}{2}mv^2 = mg \cdot DO$. The ring now begins to ascend the arc OA and its kinetic energy is expended in doing the work of raising its weight. The wire being smooth, the ring will ascend

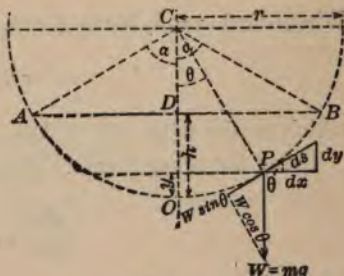


FIG. 60.

* The trajectory is so thoroughly covered by the Department of Ordnance at the Naval Academy that it is not further considered here.

to the point A , at a vertical height OD . At A the kinetic energy acquired in sliding down the opposite side, being all expended, the ring comes to rest and then starts to slide back. It will continue to oscillate. Let P be the ring's position at any time t after rest at B . Taking O for the origin and resolving the weight, mg , into components along and perpendicular to the tangent at P , the tangential component, $mg \sin \theta$, will be the accelerating force along the arc, and the equation of motion will be

$$m \frac{d^2 s}{dt^2} = m \frac{v dv}{ds} = -mg \sin \theta. \quad (1)$$

From the figure $\sin \theta = \frac{dy}{ds}$, hence (1) becomes

$$v dv = -g dy, \quad (2)$$

which integrated gives

$$\frac{1}{2} v^2 = -gy + C. \quad (3)$$

When $v=0$, $y=h$, hence $C=hg$ and (3) becomes

$$v^2 = 2g(h-y), \quad (4)$$

which agrees with (5) of Art. 128.

Using the other form of (1), noting that

$$\frac{d^2 s}{dt^2} = r \cdot \frac{d^2 \theta}{dt^2}.$$

we have

$$\frac{d^2 \theta}{dt^2} = -\frac{g}{r} \sin \theta. \quad \text{sin } \theta \approx \theta \quad \text{hence period} \quad (5)$$

Multiplying both members by $2 \frac{d\theta}{dt}$ and integrating, gives

$$\left(\frac{d\theta}{dt} \right)^2 = \frac{2g}{r} \cos \theta + C. \quad (6)$$

When $\theta = \alpha$; $\frac{d\theta}{dt} = 0$, hence $C = -\frac{2g}{r} \cos \alpha$, and (6) becomes

$$\frac{d\theta}{dt} = -\sqrt{\frac{2g}{r}} \cdot \sqrt{\cos \theta - \cos \alpha}, \text{ or } dt = -\sqrt{\frac{r}{2g}} \cdot \frac{d\theta}{\sqrt{\cos \theta - \cos \alpha}}, \quad (7)$$

the negative sign being taken as, at B , θ decreases when t increases.

Substituting the functions of the half angle, $\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2}$,

and $\cos \alpha = 1 - 2 \sin^2 \frac{\alpha}{2}$, we have

$$dt = -\frac{1}{2} \sqrt{\frac{r}{g}} \cdot \frac{d\theta}{\sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2}}}. \quad (8)$$

Let $\sin \frac{\alpha}{2} = k$ and $\sin \frac{\theta}{2} = k \sin \phi$, then

$$d\theta = \frac{2k \cos \phi d\phi}{\cos \frac{\theta}{2}} = \frac{2k \cos \phi d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

also, if $\theta = \alpha$, ϕ must equal $\frac{\pi}{2}$, and if θ is zero, ϕ must be zero, and we have

$$t = \sqrt{\frac{r}{g}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}. \quad (9)$$

The limits of (9) give us the time of $\frac{1}{4}$ of a complete oscillation; therefore,

$$T = 4 \sqrt{\frac{r}{g}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}. \quad (10)$$

Expanding, by the Binomial Theorem $(1 - k^2 \sin^2 \phi)^{-\frac{1}{2}}$, we have

$$T = 4 \sqrt{\frac{r}{g}} \int_0^{\frac{\pi}{2}} \left[1 + \frac{k^2}{2} \sin^2 \phi + \frac{1 \cdot 3}{2 \cdot 4} k^4 \sin^4 \phi + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} k^6 \sin^6 \phi + \dots \right] d\phi. \quad (11)$$

Integrating

$$T = 2\pi \sqrt{\frac{r}{g}} \left[1 + \frac{1^2}{2^2} k^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} k^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} k^6 + \dots \right]. \quad (12)$$

Substituting the value of k , gives

$$T = 2\pi \sqrt{\frac{r}{g}} \left[1 + \frac{1^2}{2^2} \sin^2 \frac{\alpha}{2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \sin^4 \frac{\alpha}{2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} \sin^6 \frac{\alpha}{2} + \dots \right]. \quad (13)$$

Equation (13) gives the time of a complete oscillation. If α is small (not exceeding 5°), all the terms in the bracket, *except the first*, may be neglected, their magnitude being so small as to be practically inappreciable. For a *small arc of swing*, the time of a complete oscillation therefore becomes

$$T = 2\pi \sqrt{\frac{r}{g}}. \quad (14)$$

132. The Simple Pendulum.—If instead of supposing the mass to move along a smooth wire, we suspend it by a weightless, inextensible cord from the point C , Fig. 60, and further, if we let the weight swing over so small an arc that its sine is not sensibly different from the angle itself, we may write equation (1) of the preceding article

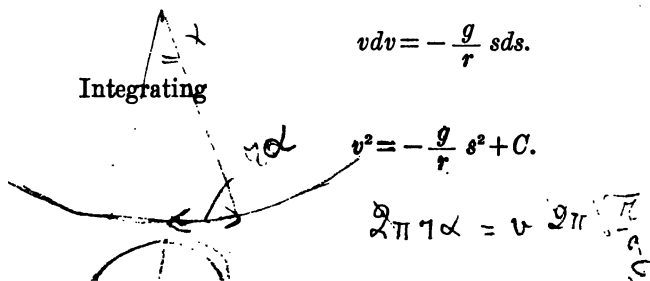
$$\frac{v dv}{ds} = -g\theta. \quad (1)$$

But the arc s is equal to $r\theta$, or $\theta = \frac{s}{r}$, hence (1) is

$$v dv = -\frac{g}{r} s ds. \quad (2)$$

Integrating

$$v^2 = -\frac{g}{r} s^2 + C. \quad (3)$$



1. Period indep. of amplitude or angle
2. max vel. depends on " " "

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When $v=0$, $s=ra$, and $C = \frac{g}{r} r^2 a^2$, and (3) is

$$\left(\frac{ds}{dt}\right)^2 = v^2 = \frac{g}{r} (r^2 a^2 - s^2) \quad (4)$$

Extracting the square root, transposing and choosing the negative sign because as t increases s diminishes, we have

$$dt = -\sqrt{\frac{r}{g}} \cdot \frac{ds}{\sqrt{a^2 r^2 - s^2}}. \quad (5)$$

Integrating

$$t = -\sqrt{\frac{r}{g}} \cdot \sin^{-1} \frac{s}{ar} + C. \quad (6)$$

When $t=0$, $s=ra$, hence $C = \frac{\pi}{2} \sqrt{\frac{r}{g}}$, and (6) is

$$t = \frac{\pi}{2} \cdot \sqrt{\frac{r}{g}} - \sqrt{\frac{r}{g}} \sin^{-1} \frac{s}{ar},$$

or, as $s=r\theta$,

$$t = \frac{\pi}{2} \sqrt{\frac{r}{g}} - \sqrt{\frac{r}{g}} \sin^{-1} \frac{\theta}{a}. \quad (7)$$

From (7), when θ is equal to zero, $t = \frac{\pi}{2} \sqrt{\frac{r}{g}}$, which is the time of one-quarter of a complete oscillation; or, for the time of oscillation,

$$T = 2\pi \sqrt{\frac{r}{g}}. \quad (8)$$

For any particular place on the earth's surface, g is a constant, therefore from (8) we may find the length of a simple pendulum, which will make a complete oscillation in one second at a place where g is known. Again, having a simple pendulum which oscillates in a second at one place, we may find the value of g at another place by observing its time of oscillation at the second

Ordinary sea-stone, unless excessive, does not affect period but only dampens the amplitude.

place. Heights of mountains, etc., may be determined by means of the pendulum.

133. Determination of Heights by Pendulum.—Actual pendulums differ from the fictitious one we have considered in that the suspending cord has weight, air resistance and friction act, and dimensions are not always accurately known. Having an actual pendulum, we may find the length, etc., of a fictitious one which has the *same period*, and this fictitious one is known as the *simple equivalent pendulum*. When experiments are made with the pendulum, the *simple equivalent* is used for the calculations. If then we transport a pendulum, which accurately oscillates in one second at the sea level where the acceleration of gravity is g , to the top of a mountain, which is so high that the variation in the force of gravity will cause a change in its time of oscillation, we may measure the height by the change caused. Calling the value of gravity at the mountain top g_1 , and referring to Art. 119, we have from (1) that the acceleration caused by gravity at the top is $g \frac{R^2}{(R+h)^2}$; therefore,

$$\frac{g}{g_1} = \frac{(R+h)^2}{R^2}, \quad \text{or} \quad g_1 = \frac{gR^2}{(R+h)^2}. \quad (1)$$

The time of oscillation on the mountain top will be, l being the length of the simple equivalent,

$$T_1 = 2\pi \sqrt{\frac{l}{g_1}} = \frac{2\pi(R+h)}{R} \sqrt{\frac{l}{g}}, \quad (2)$$

and the time of oscillation at the sea level being one second, we have

$$\frac{1}{T_1} = \frac{2\pi \sqrt{\frac{l}{g}}}{2\pi \frac{(R+h)}{R} \sqrt{\frac{l}{g}}} = \frac{R}{R+h}, \quad (3)$$

from which

$$h = R(T_1 - 1). \quad (4)$$

Problems XV.

1. Find the value of g at a place where the length of the seconds pendulum is 39.14 in. (A seconds pendulum makes a complete oscillation in two seconds.)

2. How many oscillations per minute will a pendulum, 3 ft. long, make at a place where g is equal to 32.2?

3. A small mass slides down a smooth vertical circle, starting at a point such that α is equal to 60° . What is its velocity when it reaches the lowest point of the circle?

4. A pendulum oscillates 3000 times per hour at one place and 3011 times per hour at another; compare the acceleration of gravity at the two places.

5. It is desired that the pendulum of a clock make two complete oscillations per second, how long must it be if g is 32 f/s²? $\phi = \frac{1}{2}$

6. The equation of an inverted cycloid, with the origin at the vertex, are $x = a(\psi + \sin \psi)$ and $y = a(1 - \cos \psi)$, a being the radius of the generating circle. Find the time of a complete oscillation of a small mass which slides on the curve which is considered smooth.

7. A clock gains 3 minutes a day. How much should the bob of the pendulum be raised or lowered?

8. If the radius of the earth is 4000 miles, how many minutes a day will a seconds pendulum lose if taken up a height of 5 miles?

9. If a seconds pendulum should be taken down a mine to a depth of 1260 feet, how many extra oscillations would it make in a day? see Bowser, p. 369.

10. Prove that a small mass sliding down a smooth cycloid takes the same time to reach the lowest point no matter at what place on the cycloid it is started.

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Mechanics of Sound

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In a simple pend. find ϕ where tension on string is the same as when part. hangs at rest.

$$y = \frac{2}{3} h$$

when h is the height

Feb 28
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CHAPTER XVI.

KINETIC EQUILIBRIUM.

M. 61-68

CURVILINEAR MOTION. CENTRAL FIELD.

134. Central Fields.—A *central field* of force is one in which the lines of action of the force all pass through a fixed point called the *center of force*.

The forces may be repulsive though they are usually attractive; that is, most central fields *attract*, toward the center of force, a mass at any point in the field.

The forces may vary in accordance with *any* law. Central fields in which the force varies directly as the distance from the center of force, are called *harmonic fields*.

When the force of a central field varies inversely as the square of the distance from the center of force, we have the condition of *planetary motion*.

If a mass is *at rest* when first acted upon by the force of a central field, the motion will be rectilinear, as in Chapter XIII. The motion will also be rectilinear if the mass is projected *along* one of the lines of force; that is, directly *toward* or *away from* the center of force. But a mass projected in any direction *other than* along a line of force, will move in a curvilinear path in a central field, and if the force is *attractive* the path will be concave toward the center of force.

The path of a mass moving in a central field is called the *orbit*.

The orbit must be a *plane curve* for there is no component of force acting to draw a mass out of the plane passed through the center of force and the line in which the mass enters the field.

A line from the center of force to the mass at any point of its orbit is called the *radius vector*.

135. Curvilinear Motion in a Central Field.—We will suppose a mass to be projected into an attractive central field, the force of which varies in some way with the distance of the mass from the center of force. Referring to Fig. 61, let AB be a portion of the path (orbit) described by the mass and let O be the center of force, then, if M is the position of the mass M at any time, the accelerating force will act along the radius vector, r , toward O and will be some function of r in magnitude. Taking O for the origin

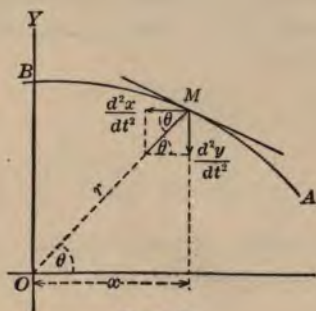


FIG. 61.

and proceeding as in Art. 75, we have, noting that r is a variable also in this case,

$$x = r \cos \theta, \quad y = r \sin \theta, \quad (1)$$

$$\frac{dx}{dt} = \frac{dr}{dt} \cos \theta - r \sin \theta \frac{d\theta}{dt}, \quad \frac{dy}{dt} = \frac{dr}{dt} \sin \theta + r \cos \theta \frac{d\theta}{dt}. \quad (2)$$

$$\left. \begin{aligned} \frac{d^2x}{dt^2} &= \frac{d^2r}{dt^2} \cos \theta - 2 \frac{dr}{dt} \sin \theta \frac{d\theta}{dt} - r \cos \theta \left(\frac{d\theta}{dt} \right)^2 - r \sin \theta \frac{d^2\theta}{dt^2} \\ \frac{d^2y}{dt^2} &= \frac{d^2r}{dt^2} \sin \theta + 2 \frac{dr}{dt} \cos \theta \frac{d\theta}{dt} - r \sin \theta \left(\frac{d\theta}{dt} \right)^2 + r \cos \theta \frac{d^2\theta}{dt^2} \end{aligned} \right\} \quad (3)$$

Now from the figure let us resolve the component accelerations, $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$, along and perpendicular to the radius vector r ; the

components are (the *outward* direction along the radius vector being regarded as positive).

$$\text{along } r; \left\{ \frac{d^2x}{dt^2} \cos \theta + \frac{d^2y}{dt^2} \sin \theta \right\} \quad (4)$$

$$\text{normal to } r; \left\{ \frac{d^2y}{dt^2} \cos \theta - \frac{d^2x}{dt^2} \sin \theta \right\}. \quad (5)$$

If now we substitute the values of $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ from (3) in (4) and (5) we will have the component accelerations along and perpendicular to r in terms of the polar co-ordinates. Reducing after this substitution, we have

$$\text{along } r; \frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2, \quad (6)$$

$$\text{normal to } r; 2 \cdot \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2} = \frac{1}{r} \cdot \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right). \quad (7)$$

Having the accelerations, we may form the equations of motion by substituting in the general formula, $F=ma$; and, using (6) and (7),

$$-m \left\{ \frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \right\} = \text{force along radius vector (negative because force acts toward center)}. \quad (8)$$

$$\frac{m}{r} \cdot \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) = \text{force normal to } r = \text{zero}. \quad (9)$$

Equations (8) and (9) are the general equations of motion in a central field. Equation (9) is always equal to zero in a central field (the field alone being considered) for the reason given in the preceding article; i. e., the force, at all points of the field, acts *along* the radius vector toward the center of force having no component normal to r . This being the case we will integrate (9), getting

$$r^2 \frac{d\theta}{dt} = 0 + C. \quad (10)$$

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$$v^2 = h^2 \left(\frac{du}{dr} \right)^2$$

also when force always crosses same axis

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Now observing that the element of area described by the radius vector in unit time is $\frac{dA}{dt} = \left(r \frac{d\theta}{dt} \cdot \frac{r}{2} \right)$, which is equal to one-half the left member of (10), and also noting that by (10) this value of $\frac{dA}{dt}$ is a constant, we may, if we call this constant h , write (10)

Now $r^2 \frac{d\theta}{dt} = h = 2 \cdot \frac{dA}{dt}$ Integral of $\frac{1}{r^2}$ areas:

$$\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{h}{r^2} \cdot \frac{dr}{d\theta}; \quad xy' - yx' = A$$

and if we put $r = \frac{1}{u}$, we have from (11)

$$\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2. \quad (12)$$

Also

$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \frac{d\left(\frac{1}{u}\right)}{d\theta} \cdot hu^2; \text{ or, } \frac{dr}{dt} = -\frac{1}{u^2} \left(\frac{du}{d\theta}\right) hu^2 \\ &= -h \frac{du}{d\theta}. \end{aligned} \quad (13)$$

and

$$\begin{aligned} \frac{d^2r}{dt^2} &= -h \frac{d}{dt} \left(\frac{du}{d\theta}\right) = -h \frac{d}{d\theta} \left(\frac{du}{d\theta}\right) \frac{d\theta}{dt} \\ &= -h \frac{d^2u}{d\theta^2} \cdot \frac{d\theta}{dt} = -h^2 u^2 \frac{d^2u}{d\theta^2}. \end{aligned} \quad (14)$$

Denoting by F_r the force along the radius vector at any instant, we have from (8)

$$F_r = m \left\{ r \left(\frac{d\theta}{dt}\right)^2 - \frac{d^2r}{dt^2} \right\}$$

from which by substitution from (12) and (14), we have

$$F_r = mh^2 u^3 \left(\frac{d^2u}{d\theta^2} + u \right). \quad (15)$$

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Equation (15) is known as the differential equation of the orbit.

It will be noted that by equation (11) the area swept over by the radius vector in a unit of time is *constant*; also, the angular velocity, $\frac{d\theta}{dt}$, is inversely proportional to the square of the distance of the mass from the center of force.

From equation (15) it is clear that, knowing the law governing the force in a central field, the orbit may be found; or conversely.

136. To Find the Orbit in an Attractive Central Field, the force being inversely proportional to the squared distance from the center of force.—When the force is inversely proportional to the square of the distance, the total force acting on a mass m is $\frac{k^2m}{r^2}$, or k^2mu^2 , when $u = \frac{1}{r}$. Substituting in equation (15), Art. 135, we have

$$\frac{d^2u}{d\theta^2} + u = \frac{k^2}{h^2}. \quad (1)$$

Letting $u_1 = u - \frac{k^2}{h^2}$, and substituting in (1), gives

$$\frac{d^2u_1}{d\theta^2} + u_1 = 0. \quad (2)$$

Let $p = \frac{du_1}{d\theta}$, then $\frac{dp}{d\theta} = \frac{d^2u_1}{d\theta^2}$, and multiplying (2) by $2p$, we

$$2p \cdot dp + 2u_1 du_1 = 0, \quad (3)$$

which integrated gives $p^2 + u_1^2 = c_1^2$ or $\left(\frac{du_1}{d\theta}\right)^2 + u_1^2 = c_1^2$, from which

$$\frac{du_1}{\sqrt{c_1^2 - u_1^2}} = d\theta. \quad (4)$$

Integrating (4) we have

$$-\cos^{-1} \frac{u_1}{c_1} = \theta + c_2; \text{ or, } \cos^{-1} \frac{u_1}{c_1} = -(\theta + c_2), \quad (5)$$

or

$$\frac{u_1}{c_1} = \cos(\theta + c_2). \quad (6)$$

Substituting for u_1 and transposing,

$$\frac{1}{r} = u = c_1 \cos(\theta + c_2) + \frac{k^2}{h^2}. \quad (7)$$

Equation (7) is the polar equation of a conic section, the focus being the pole (see conic sections); therefore, masses projected into a central field, the force of which is inversely proportional to the square of the distance from the center of force, will move in orbits which are conic sections having the center of force at a focus.

The conic section is an ellipse, parabola or hyperbola, according as the eccentricity is less than, equal to, or greater than unity. Now the general equation of a conic section, whose axis makes an angle α with the initial line and whose origin is at a focus, is

$$r = \frac{ep}{1 + e \cos(\theta - \alpha)}, \quad (8)$$

where p is the distance from the focus to the directrix.

Equation (7) reduces to the form

$$r = \frac{\frac{h^2}{k^2}}{1 + c_1 \frac{h^2}{k^2} \cos(\theta + c_2)}, \quad (9)$$

therefore equation (7) is a conic section, the eccentricity of which is $c_1 \frac{h^2}{k^2}$, and the principal axis of which makes an angle of $-c_2$ with the initial line of the conic. The distance from the focus to

the directrix is $\frac{1}{c_1}$. A circle is an ellipse whose foci are coincident and whose directrix is at an infinite distance; for a circular orbit then the value of c_1 must be zero and, from (9) r must equal $\frac{h^2}{k^2}$.

137. Kepler's Laws.—After an extended series of observations of his own, and a study of the records of other observers, the astronomer Kepler, in the seventeenth century, pronounced the following to be the laws of planetary motion.

I. The orbits of the planets are ellipses with the sun at a focus.

II. With the sun as origin, the radius vector of each planet sweeps over equal areas in equal times.

III. The squares of the periods of revolution of each planet are as the cubes of the major semi-axes of the orbits.

Later Newton proved that the force of the planetary central field varied inversely as the square of the distance from the sun.

138. Newton's Law of Universal Gravitation.—By this law: *Every mass, or particle of matter in the universe attracts every other mass, or particle, with a force which varies directly as the product of the masses, inversely as the square of the distance between the masses, and which acts along the straight line connecting the centers of the masses.*

Observation shows that the moon, and other satellites, follow laws corresponding to those of Kepler for the planets; the primary (the earth is primary to the moon), occupying the focus of the satellite's orbit.

The slight deviations from the orbits deduced under Kepler's laws, which observations show to exist in the motions of the planets and their satellites, may be traced to Newton's law of universal gravitation.

$$x'' = -\frac{P}{r^2} \quad y'' = -\frac{P}{r^2} \frac{y}{x}; \text{ hence } xy' - yx' = h = r^2 \phi'$$

from Calc. $x dy - y dx = p ds$ Hence $v = \frac{h}{p}$

$n = \frac{ds}{dt} = \frac{h}{p}$

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Problems XVI.

1. Prove that at any point in its orbit the speed of a mass moving in a central field is inversely proportional to the perpendicular distance from the center of force to the tangent to the orbit at the point.

2. Show that the angular velocity of a mass which moves in a circle of given radius in a central field, the center of force being at the center of the orbit, is constant. $r^2 \phi' = h$, also $\phi' = \frac{v^2}{r}$

3. A mass moves in an attractive central field of force which varies inversely as the 5th power of the distance. Prove that the orbit, which passes through the center of force, is a circle.

4. Prove that the attraction under which a mass will describe the orbit, $r = 2a \cdot \cos \theta$, in a central field, is $\frac{8a^2 h^2 m}{r^5}$.

5. If the orbit of a planet has a major semi-axis four times that of the earth, prove that its period is eight times that of the earth.

6. When a mass describes the hyperbola, $r = \frac{a(e^2 - 1)}{1 + e \cdot \cos \theta}$, prove that the attraction is $\frac{h^2 m}{a(1 - e^2)r^2}$.

7. The angular advance of the moon in its orbit is 31'.5 per hour. If the mass of the moon is 1/81 that of the earth and its distance from the earth's center equal to 60 of the earth's radii (4000 miles), prove its period is about 29 days.

8. If a mass describes an ellipse in an attractive central field, the center of force being at the center of the ellipse, prove that the force varies directly as the distance, and is equal to $\frac{h^2 r}{a^2 b^2}$, a and b being the semi-major and semi-minor axes of the ellipse.

9. If the center of force, problem 8, had been at one of the foci, prove that the velocity at any point of the orbit is equal to

$$\sqrt{\frac{h^2(2a - r)}{a^2 r(1 - e^2)}}.$$

10. Prove that in a central field the attractive force that will make the orbit of a mass a parabola with the center of force at the focus, is $\frac{h^2 m}{2ar^3}$. The equation of a parabola with the origin at the focus is $r = \frac{2a}{1 + \cos \theta}$.

11. The Earth's path abt Sun being assumed to be a circle, show that the longest time that a comet, which describes a parabolic path, can remain within the Earth's orbit is $\frac{2}{3\pi}$ years.

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CHAPTER XVII.

KINETIC FRICTION.

Sl. 178-231

139. Kinetic Friction.—Kinetic friction may be defined as the friction between masses which move relatively to each other.

The laws of friction, given in Art. 36, which are based entirely upon experiments, hold for kinetic friction as well as for statical, but the coefficient of kinetic friction should be a little less than that for statical, because, after the start, the motion itself tends to smooth the rubbing surfaces (or one of them), also air gets between the surfaces and acts as a lubricant; therefore, the friction, *after motion exists*, will be a little less than the limiting statical friction. However, as the coefficient, in both statical and kinetic friction, is the ratio between the friction force and the normal pressure, the laws, as stated in Art. 36, will hold in both for ordinary speeds. Actually, kinetic friction varies a trifle with the velocity and the lapse of time, becoming less as the velocity or time increases, but the laws governing these changes are not known, though, from the results of experiments, the facts are apparent.

The laws being the same, the friction for bodies dragged along horizontal or inclined planes, in fact, that for any rectilinear motion where the mass moving *slides* along, is found by the same methods as those already described in statical friction, the difference in *amount* being due to the lesser value of the coefficient used for kinetic friction. But, rolling friction, and the friction of bearings, pivots, etc., must be handled in a different way and these will be the subjects considered in this chapter.

140. The Moment of Friction.—When a mass slides along a surface the friction, at each point of contact, is directly opposite

to the motion and the lines of action of the friction for all points of contact, are parallel, the friction all acting in the same direction. If, however, the surface of contact is rotating about an axis perpendicular to its plane, the lines of action of the friction at the different points of contact will not be parallel. The friction at each point will act in a direction opposite to that of the motion of the point, and the friction at a point *close* to the center of motion will have less effect in retarding the motion than that at a point further away from the center of motion (principle of the lever). Therefore, to get the retarding effect over the whole surface, we must add together the retarding effects at each point of the surface of contact. In other words, we must find the sum of the moments, about the center of motion, of the friction at all the points of contact, and this sum will be equal to the moment, about the center of motion, of that part of the turning force which is used up in overcoming the frictional resistance.

To illustrate, suppose a vertical, circular shaft of weight W , turns about its axis and is supported in a horizontal, flat bearing; if a is the radius of the shaft, the pressure on each unit area of the bearing surface will be $\frac{W}{\pi a^2}$. Let μ be the coefficient of friction.

Then the moment of the friction on any element of area dA , which is all at a distance r from the center of motion, will be $r \cdot \frac{\mu W \cdot dA}{\pi a^2}$;

which, integrated between the proper limits, gives the moment of the friction acting on the bearing. In the shaft above, the most convenient element of area dA , which is all at a distance r from the center of motion, is the circumferential element, $2\pi r dr$; therefore, calling the moment M ,

$$M = \int_0^a \frac{\mu W r}{\pi a^2} \cdot 2\pi r dr = \frac{2}{3} \cdot \mu W a. \quad (1)$$

The total friction is equal to μW ; therefore, the mean arm for the friction, in *this* case, is $\frac{2}{3}a$; that is to say, if the whole weight

rested on a circumference of radius $\frac{3}{2}a$, it would cause a frictional moment equal to the above.

141. Moment of the Friction of a Conical Pivot.—Let Fig. 62 represent the section of a pivot in a conical seat. Then if p is the normal pressure on a unit of area of the inclined surface, the sum of all the vertical components of p must be equal to W , the vertical force with which the pivot is pressed into its bearing. Or, if A_c is the area of the curved surface, then

$$pA_c \sin \phi = W. \quad (1)$$

Now the area of the base of a right cone is equal to the product of the inclined surface and the sine of half the vertical angle, or

$$A_c \sin \phi = \pi a^2. \quad (2)$$

Therefore, from (1),

$$p = \frac{W}{A_c \sin \phi} = \frac{W}{\pi a^2}. \quad (3)$$

from which it is seen that the normal pressure p is independent of the angle of the cone.

The friction on each element of the inclined area will be $\mu p dA$, and if the element is at a distance x from the axis of rotation, the moment of the friction will be $x \cdot \mu p dA$. Taking a circumferential element, its area will be $2\pi x ds$ (see figure); but

$$\frac{x}{y} = \frac{a}{a \cot \phi},$$

or $y = x \cot \phi$, and $dy = dx \cot \phi$, $\therefore ds = \sqrt{dx^2 + dy^2} = \text{cosec } \phi \cdot dx$, and the element of area is $2\pi \text{ cosec } \phi \cdot x dx$. Substituting these

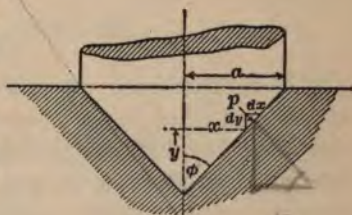


FIG. 62.

values and that of p from (3), in the expression for the moment of the friction on an element, and integrating, we have

$$M = \frac{2\mu W}{a^2 \sin \phi} \int_0^a x^2 \cdot dx = \frac{2}{3} \cdot \frac{a\mu W}{\sin \phi}. \quad (4)$$

Recent experiments show this value of the moment of the friction for a conical pivot to be a trifle large, the experimental value lying between $\frac{1}{2}$ and the coefficient $\frac{2}{3}$ as above. The lesser experimental value may be due to the falling off in the friction on account of the velocity or to the fact that it is very difficult to get perfect conical bearing surfaces.

To find the work lost by friction, the moment of the friction is the force multiplied by the mean arm through which the force is turned; also, the space through which the friction is overcome during one revolution is 2π multiplied by the mean arm; therefore, the work done against friction during one revolution is M multiplied by 2π , or the work lost by overcoming friction, during one turn of a conical pivot, is

$$\text{Work lost} = 2\pi M = \frac{4\pi a\mu W}{3 \sin \phi} \text{ foot-pounds.} \quad (5)$$

If the pivot makes n revolutions per minute, the horse-power lost by overcoming friction, is

$$\text{HP lost} = \frac{4\pi a n \mu W}{3 \sin \phi \cdot 33000}. \quad (6)$$

142. Belt Friction. Strap Friction Brakes, etc.—The friction of a belt on a flat pulley or wheel is found in Art. 40, for though the wheel and belt are both in motion, they have no velocity relative to each other as long as the combination is working properly. The formula is, calling T_1 the tension of the belt just as it leaves the wheel, T_2 the tension just as it comes in contact with the wheel, μ the coefficient of friction and θ the angular measure of the arc over which the belt is in contact (e being the base of the Napierian system of logarithms)

$$T_1 = T_2 e^{\mu \theta}, \quad (1)$$

or arranged for computation with common logarithms,

$$\log \frac{T_1}{T_2} = .4343 \mu \theta. \quad (2)$$

Formula (2) may be used for finding the friction of a rope wound round a post; of strap friction brakes where the strap circles the friction arc; and of belts used for the transmission of power.

To find the power transmitted by a belt; the difference in tension between the part of the belt just leaving the wheel and that part just coming into contact with the wheel, measures the friction. A wheel which makes n revolutions per minute, causes a belt, which does not slip, to travel $2\pi rn$ f/m, therefore the work done (force times space) is $2\pi rn(T_1 - T_2)$ foot-pounds per minute, or as $2\pi rn$ is the linear velocity, v ,* of the belt, the work is $V(T_1 - T_2)$ foot-pounds per minute; or,

$$HP = \frac{2\pi rn(T_1 - T_2)}{33000} = \frac{v(T_1 - T_2)}{33000} \quad (3)$$

143. Rolling Friction.—When a wheel or cylinder rolls along a plane the resistance to its motion is called rolling friction or rolling resistance. The amount of the rolling resistance (which is usually less than sliding friction) depends upon the softness of the plane and the radius and width of tire of the wheel or cylinder.

Suppose a wheel, of radius r , rolls along a plane as shown in Fig. 63. Let the weight be W and let a horizontal force P , acting at the axle, be required to pull the wheel along at a uniform velocity; then, W

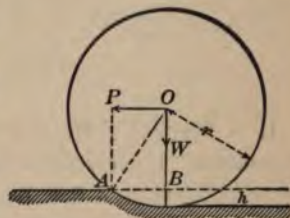


FIG. 63.

* v is given in feet per minute in this formula because n is in revolutions per minute.

being vertical and P horizontal, the vertical and horizontal components of the resistance of the plane must be equal respectively to W and P ; therefore, P is a measure of the rolling friction. Now, if the lowest point of the wheel sinks a distance h below the original level of the plane, leaving a track h deep, it is clear that the resistance is due to the wheel being continually pulled over an obstruction at A . Taking moments about A and thus eliminating the unknown resistance of the plane which acts through that point, we have

$$P \times BO = W \times AB. \quad (1)$$

But BO is equal to $r-h$, hence

$$P = \frac{W}{r-h} \cdot AB. \quad (2)$$

Compared with r the value of h is very small, therefore we may write for ordinary roads

$$P = \frac{W}{r} \cdot AB. \quad (3)$$

It is clear that the length of AB will depend both upon the weight and the width of tire, as stated above. Like the coefficient of friction the value of AB must be determined by experiment.

Problems XVII.

1. A vertical shaft is formed of two equal circular cylinders in contact. Each weighs 500 lbs. and is 6 in. in diameter. They are supported on a flat, horizontal bearing and revolve about the line of contact. If $\mu = \frac{7}{22}$, find the moment of the friction.

2. Show that the moment of the friction of a square prism, of weight W , which rotates about its vertical axis, and is supported on a flat, horizontal bearing, varies as the side of the square.

3. A vertical shaft, radius 2 in., weight 2500 lbs., is supported on a conical pivot, the vertical angle of which is 60° . What horse-

power is lost by friction if the shaft makes 140 revolutions per minute? $\mu = .08$.

4. If the shaft of problem 3 had been on a flat, horizontal bearing, what horse-power would have been lost by friction?

5. The loss of horse-power by friction on a flat, horizontal bearing is .36 HP. If the bearing is 2 in. in radius and the shaft is making 100 revolutions per minute, what is the pressure? $\mu = .06$.

6. A brake is pressed against a car-wheel with a force of 5 tons, $\mu = .3$. Find the horse-power lost in overcoming friction when the axle of the wheel is moving at 20 m/h.

7. A horizontal shaft, 3 in. in radius, exerts a pressure of 3 tons on its bearing. If $\mu = .012$ and the shaft is making 80 revolutions per minute, find the horse-power lost at the bearing.

8. A weight of 5 tons is lifted by a rope wrapped $3\frac{1}{2}$ times round a drum. What pull on the part of the rope leading from the drum is required?

9. A belt traveling at 700 f/m and in contact with half the circumference of the driving wheel, transmits one horse-power. Find its tension at the points where it comes in contact with and where it leaves the wheel. $\mu = .25$.

10. The tension of a belt on the slack side is one-half its tension on the taut side. The limiting tension per inch of width is 40 lbs. How wide must the belt be to transmit 40 horse-power over a wheel 3 ft. in diameter, if the wheel is making 100 revolutions per minute?

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CHAPTER XVIII.

KINETIC EQUILIBRIUM.

H. 36-44
240-242

VIBRATORY MOTION OF RIGID BODIES ABOUT FIXED AXES.
COMPOUND AND BALLISTIC PENDULUMS.

144. General Conditions.—When a rigid-body moves about a fixed axis the accelerating forces, inertia forces, etc., are changing their directions continually. Each separate particle of the rigid body will be moving in a circular path and the forces which act on it, act through its center of gravity, and are therefore always at a fixed distance (the radius of the path) from the center of motion. It is more convenient to use the *moments* of the forces about the center of motion than to proceed by the methods used heretofore. Therefore, when considering the motion of a rigid body about a fixed axis passing through itself, we will use for the equation of motion, *the moment of the force is equal to the moment of the mass times the acceleration*; and will call this equation the *moment equation*.

To illustrate, a particle of mass m is at a distance r from the axis about which a rigid body rotates. At the *instant* under consideration, the particle (and rigid body also) is moving with an angular velocity, $\tilde{\omega} = \frac{d\theta}{dt}$. The linear velocity, v , is then $r\tilde{\omega} = r \frac{d\theta}{dt}$; the angular acceleration is, $\beta = \frac{d\tilde{\omega}}{dt} = \frac{d^2\theta}{dt^2}$, and the linear acceleration is $a = r \frac{d\tilde{\omega}}{dt} = r \frac{d^2\theta}{dt^2}$.

The equation of motion that we have been using would be

$$\text{Force} = ma = mr \frac{d\tilde{\omega}}{dt} = mr \frac{d^2\theta}{dt^2}. \quad (1)$$

Equation (1) is true for the force acting *at the instant* on the particle; but the *next* instant, even though the force is the same, its direction will have changed. The *turning effort* of the force, in urging the particle to move about the fixed axis, will be in the same direction at whatever point of the circular path we consider it; therefore, if we use the *moment* of the force, we can, without error, add together the moments of the forces acting on the particle at *all* points of its circular path, and the sum will be the moment of the sum of all the forces which have acted on it during its revolution. Therefore the moment equation is

$$\left. \begin{array}{l} \text{moment of} \\ \text{force} \end{array} \right\} = \left. \begin{array}{l} \text{moment of} \\ ma \end{array} \right\} = mr^2 \frac{d\tilde{\omega}}{dt} = mr^2 \frac{d^2\theta}{dt^2}. \quad (2)$$

It will be noted that mr^2 is the "moment of inertia of the mass" (see Art. 105) of the particle about the fixed axis, and that $\frac{d^2\theta}{dt^2} = \beta$ = angular acceleration; therefore, from (2) we may state that the moment of the force acting on the particle is equal to the "moment of inertia of its mass" times the angular acceleration. If we add together the moments of these forces and the moments of the inertia for *every* particle in the rigid body, we have

$$\text{Moment of } F = I\beta = I \frac{d^2\theta}{dt^2}. \quad (3)$$

Equation (3) is the *general moment equation* for rigid bodies moving about a fixed axis through themselves. I is the moment of inertia of the whole mass of the rigid body about the fixed axis. F is the force which causes the motion and it may be applied at any point of the rigid body; therefore, care must be taken to use, for the moment of F , the product of F (the impressed force)

and the distance from the fixed axis to the point of application of F .*

145. Kinetic Energy of Rigid Bodies Moving About Fixed Axes.—Consider a particle at a distance r from the fixed axis about which a rigid body is rotating. At any instant when the angular velocity is $\tilde{\omega}$, the linear velocity of the particle is $r\tilde{\omega} = r \frac{d\theta}{dt}$, and its kinetic energy at the instant will be

$$\frac{1}{2}mv^2 = \frac{1}{2}mr^2\tilde{\omega}^2 = \frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2.$$

The moment of inertia of the mass of the particle about the fixed axis is mr^2 ; therefore, as adding together the kinetic energies of *all* the particles in the rigid body gives the kinetic energy of the whole body, we have

$$\text{K. E.} = \frac{1}{2}I\tilde{\omega}^2 = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2. \quad (1)$$

It will be noted that the *form* of this equation is the same as that for the kinetic energy in rectilinear motion.

146. Comparing Formulæ of Translation and Rotation.—

Translation.

$$v = v_0 + at.$$

$$s = s_0 + v_0t + \frac{1}{2}at^2.$$

$$F = ma = m \frac{d^2s}{dt^2}.$$

$$\text{K. E.} = \frac{1}{2}mv^2.$$

Rotation.

$$\tilde{\omega} = \tilde{\omega}_0 + \beta t.$$

$$\theta = \theta_0 + \tilde{\omega}_0t + \frac{1}{2}\beta t^2.$$

$$\text{Moment of } F = rm(r\beta) = I \frac{d^2\theta}{dt^2}.$$

$$\text{K. E.} = \frac{1}{2}I\tilde{\omega}^2.$$

147. Problem on Motion About a Fixed Axis.—Let a thin straight rod, a long and of uniform cross-section, be hinged

* All forces which act on a body, other than the internal forces acting among its particles, are called *impressed forces*; or, impressed forces are all *external* forces acting on a body. Gravity is an impressed force, it being the earth's attraction for the body.

(smooth hinge) at one end so that it may turn freely in the vertical plane. Let Fig. 64 represent such a rod. If the rod is raised to a horizontal position and released, gravity will cause it to turn about the point O . The center of gravity of the rod is at its middle point. Let W be its weight. When the rod, after release, has turned through the angle $(90^\circ - \theta)$ its weight has moved through a vertical space equal to $\frac{a}{2} \cos \theta$, and work to

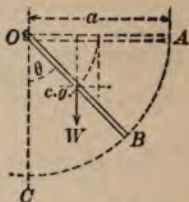


FIG. 64.

the amount of $W \frac{a}{2} \cos \theta$, has been done.

Everything being smooth, *all* this work has been done in giving angular velocity to the rod, therefore, the kinetic energy possessed by the rod at the instant it makes the angle θ with the vertical is equal to the work done on the rod, hence the equation of energy is

$$\frac{1}{2} I \bar{\omega}^2 = W \frac{a}{2} \cos \theta. \quad (1)$$

The moment of inertia of the mass of a uniform rod about a perpendicular axis through its end is $\frac{a^2 m}{3} = \frac{a^2 W}{3g}$, therefore (1) is

$$\frac{a^2 W}{6g} \bar{\omega}^2 = \frac{1}{2} W a \cos \theta, \quad (2)$$

from which

$$\bar{\omega}^2 = \frac{3g \cos \theta}{a}. \quad (3)$$

Equation (3) gives the angular velocity for any value of θ ; for example, $\theta = 60^\circ$, $\bar{\omega} = \sqrt{\frac{3g}{2a}}$; $\theta = 0^\circ$, $\bar{\omega} = \sqrt{\frac{3g}{a}}$; $\theta = -90^\circ$, $\bar{\omega} = 0$.

The above is the simplest way of getting the velocity; for the acceleration, it is best to substitute in the *moment equation*. Noting that the part of the weight (the rod being at any angle θ)

which is normal to the rod; *i. e.*, the part which causes angular acceleration, is $W \sin \theta$, the moment equation is *

$$-W \frac{a}{2} \sin \theta = I \frac{d^2\theta}{dt^2} = \frac{Wa^2}{3g} \cdot \frac{d^2\theta}{dt^2}, \quad (4)$$

transposing

$$\beta = \frac{d\ddot{\omega}}{dt} = \frac{d^2\theta}{dt^2} = -\frac{3g \sin \theta}{2a}. \quad (5)$$

Equation (5) gives the angular acceleration when the rod makes any angle θ with the vertical; for example, $\theta = 60^\circ$, $\frac{d^2\theta}{dt^2} = \frac{3\sqrt{3} \cdot g}{4a}$; $\theta = 0$, $\frac{d^2\theta}{dt^2} = 0$; $\theta = -90^\circ$, $\frac{d^2\theta}{dt^2} = -\frac{3g}{2a}$. The latter is the value of the angular acceleration when starting from rest, the sign being changed. Equation (5) may also be found by differentiating equation (3) with respect to t .

148. The Compound Pendulum.—We have investigated the motion of the simple pendulum, an ideal arrangement which it is impossible to contrive practically, as it would have to be a volumeless mass suspended by a weightless cord. Actual pendulums usually consist of a bob hung by a small rod from a fixed point. They are readily adjusted to oscillate in the proper time by raising or lowering the bob which is on a screw thread. Such pendulums are known as compound pendulums.

Every compound pendulum has its simple equivalent (see Art. 133); that is, the length of the ideal simple pendulum, which will oscillate in the same time as the compound pendulum, is found and used in such calculations as involve the length, the time being measured by the compound pendulum. If the length of the simple equivalent pendulum be measured off, from the point of suspen-

* The component of the weight which acts along the rod toward the free end ($W \cos \theta$) is balanced by the resistance furnished by the hinge. This component makes no change in the angular acceleration as the hinge is smooth.

sion, on the compound pendulum and the compound pendulum suspended from this new point, the time of oscillation will be the same as it was before the change. This new point of suspension is called the *center of oscillation*, and it always lies below the center of gravity of the compound pendulum. It is the point where the mass of the simple pendulum would have to be concentrated for the periods of both pendulums to be equal. Suppose Fig. 65 represents a compound pendulum suspended from the point O . Let its mass be $m = \frac{W}{g}$, and let

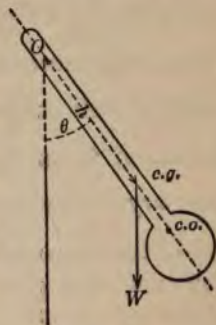


FIG. 65.

$I = k^2 m$ be the moment of inertia of its mass about an axis through O perpendicular to the plane of the paper, also let the distance from the center of gravity to the point of suspension be h , then, as in the preceding article, the moment equation is

$$-Wh \sin \theta = k^2 m \frac{d^2 \theta}{dt^2}. \quad (1)$$

The minus sign is used as θ decreases when t increases. Transposing

$$\frac{d^2 \theta}{dt^2} = - \frac{gh \sin \theta}{k^2}. \quad (2)$$

Equation (2) is of the same form as equation (5) of Art. 131, and when integrated for the time, gives

$$T = \frac{2\pi k}{\sqrt{gh}} \left[1 + \left(\frac{1}{2} \right)^2 \sin^2 \frac{\alpha}{2} + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \sin^4 \frac{\alpha}{2} + \dots \right]. \quad (3)$$

for the time of a complete oscillation, α being the initial angle to the vertical. This result is identical with that of Art. 131 if we let $r = \frac{k^2}{h}$, the r representing the length, l , of the simple pendu-

lum. This may be readily shown if we assume that the oscillations are small. In that case the period of the compound pendulum is $T = 2\pi \sqrt{\frac{k}{gh}}$; that of the simple pendulum is $T = 2\pi \sqrt{\frac{l}{g}}$. These times must be equal for the simple equivalent; therefore, equating their values and solving for l , we have, $l = \frac{k^2}{h}$. It is convenient to use this simple equivalent when calculations are necessary. If we measure off a distance, $l = \frac{k^2}{h}$, from O along the axis of the compound pendulum, we will find the center of oscillation (marked *c. o.* in the figure). Having the moment of inertia of the mass about a parallel axis through the center of gravity, say $k_0^2 m$, we have $k^2 = k_0^2 + h^2$, or in terms of k_0 , $l = \frac{k_0^2 + h^2}{h}$; transposing this last equation, gives $(l - h)h = k_0^2$, which shows that the point of suspension may be changed to the center of oscillation without changing the period.

149. Ballistic Pendulum.—An important application of the theory of the compound pendulum is an arrangement, called the ballistic pendulum, for determining the velocity of small projectiles. The ballistic pendulum consists of a heavy mass (a large box of sand or a block of wood), suspended from knife edges so that it may swing freely in the vertical plane. The pendulum being at rest, a projectile is fired into it, remaining embedded in the mass, but causing the mass to swing. The angular velocity of the pendulum may be reduced by increasing the mass, and vice versa. Calling M the mass of the pendulum with the projectile in it, m the mass of the projectile, v its striking velocity and b the distance from the point, or axis, of suspension to the point of impact of the projectile, the remaining notation being as in the preceding article, we will proceed to find the striking velocity of the projectile.

The impulsive force of the projectile, when it strikes the mass, is measured by its momentum, Art. 97, and the momentum, mv , of the projectile before impact must be equal to the momentum of the mass M after impact (not taking account of losses by heat, friction, etc.). For any angular motion about the axis of suspension the whole mass M must be considered as concentrated at a distance k , the radius of gyration, from the center of motion (Art. 106); therefore, the linear velocity of the mass m is $\tilde{\omega}k$ ($\tilde{\omega}$ being its angular velocity), its momentum is $M\tilde{\omega}k$ and the moment of its momentum about the center of motion is $k \cdot M\tilde{\omega}k$. The moment of the momentum of the projectile about the axis of suspension is bmv . Equating these two values of the moment of the momentum, we have

$$bmv = k^2 \tilde{\omega} M, \quad \text{or} \quad \tilde{\omega} = \frac{bmv}{k^2 M}. \quad (1)$$

From (2), of Art. 148, the moment equation of the compound pendulum is

$$\frac{d^2\theta}{dt^2} = -\frac{gh \sin \theta}{k^2}. \quad (2)$$

Integrating (2) by the method of Art. 131, we have

$$\left(\frac{d\theta}{dt}\right)^2 = \tilde{\omega}^2 = \frac{2gh}{k^2} (1 - \cos \alpha) = \frac{4gh}{k^2} \sin^2 \frac{\alpha}{2}. \quad (3)$$

α being the angle of swing, out from the vertical, of the ballistic pendulum.

Equating the values of $\tilde{\omega}$ from (1) and (3), we have

$$v = \frac{2kM\sqrt{gh}}{bm} \sin \frac{\alpha}{2}. \quad (4)$$

Equation (4) gives the striking velocity.

The muzzle velocity may be determined by suspending the *gun* so that it will swing when it receives the shock of the recoil.

angle
150. Torsion.—A twisted wire will exert an effort to untwist that will equal the effort necessary to twist it (Newton's third law). Also, by Hooke's law (Art. 99) the twisting effort is proportional to the amount of twist. The word effort in each case above, meaning the *moment*, about the axis of twist, of the force which causes or resists the motion.

If we let f be the force, in pounds, which, applied at a distance a feet from the axis of twist, will turn a wire (Fig. 66), which is held fixed at the upper end, through one radian at its lower end; then, the torque, or moment of the force per radian of twist, is af in pounds-feet units. This torque is resisted by an equal torque which tends to untwist the wire, and which, if the force f be removed, will do so, giving the wire a certain angular acceleration about the axis of twist. Calling this angular acceleration,

$\beta = \frac{d^2\theta}{dt^2}$, and the moment of inertia of the mass of the wire about its geometrical axis (the axis of twist) I , we have the moment equation, remembering that by Hooke's law the force varies directly with the angle of twist,

$$I \frac{d^2\theta}{dt^2} = -fa\theta, \quad (1)$$

the sign being negative because when released the resisting torque decreases θ as t increases.

Transposing and integrating, as in Art. 132, we have, after extracting the square root,

$$\frac{d\theta}{dt} = -\sqrt{\frac{fa}{I}} \cdot \sqrt{a^2 - \theta^2}, \quad (2)$$

where a is the whole angle through which the wire was twisted.

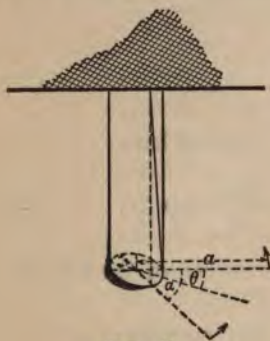


FIG. 66.

Integrating for time, as in Art. 132, we have

$$t = \sqrt{\frac{I}{fa}} \cdot \left\{ \frac{\pi}{2} - \sin^{-1} \frac{\theta}{a} \right\}. \quad (3)$$

Equation (3) is of the same form as (7), of Art. 132, hence the motion is vibratory, and the time of a complete oscillation is

$$T = 2\pi \sqrt{\frac{I}{fa}} = 2\pi \sqrt{\frac{I}{M}}, \quad (4)$$

where M is the absolute torque.

In using this formula the torque for particular lengths of wire must be determined, M being the torque in pounds-feet units per radian of twist for any particular wire. 1. Theorem for mom. of in.

Problems XVIII. 2. Rigid suspension

1. A thin rod of uniform cross-section and 10 ft. long, oscillates about an axis through one end. What is its period?

$$T = 2\pi \sqrt{\frac{2a}{3g}}$$

2. Find the radius of gyration of a mass, suspended from its end, which makes 36.5 complete oscillations per minute. The center of gravity of the mass is 38 in. from the point of suspension.

$$T = 2\pi \sqrt{\frac{k}{gh}}$$

3. A thin rod, 5 ft. long and of uniform cross-section, is hinged at one end. It is turned to a vertical position above the hinge and released. Find the velocity of the free end (1) when it is horizontal, (2) when it is vertical, below the hinge.

4. A thin circular plate of radius a , oscillates about an axis perpendicular to its plane through a point in its circumference. Find the length of the simple equivalent pendulum.

5. A cylindrical bar of $1\frac{1}{2}$ in. radius and $1\frac{1}{2}$ ft. long is suspended from an axis which coincides with a diameter of one of its ends. If it is slightly disturbed, how many complete oscillations will it make in a minute?

6. A straight piece of wire, b long, is stood vertically on end and allowed to fall over. What is its kinetic energy when it strikes the ground?

at what pt is the period same?

7. A uniform rod, 3 ft. long, oscillates about a horizontal axis which is 1 ft. from its center of mass. Find the length of its simple equivalent pendulum.

8. A circular plate, 5 in. radius, weighs 6 lbs. It hangs with its plane horizontal by a thin wire fastened to its center. A moment of 1 lb.-ft. turns the plate through 10° . How many oscillations will it make per minute?

9. What must be the ratio between the radius of the base and the altitude of a right cone, that the center of oscillation may be in the base when it is suspended at the vertex?

10. A wire, 10 ft. long, is bent into a cycloid. If it oscillates about a horizontal axis between its ends, what is the length of its simple equivalent pendulum?

Additional on comp. pend.

" " diff. eqn p. 203 & 206

" " tract. determ. mom. of in.

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CHAPTER XIX.

KINETIC EQUILIBRIUM.

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COMPLETE ROTATION OF RIGID BODIES ABOUT FIXED AXES.

FLY-WHEEL, CYLINDERS, ETC.

151. Complete Rotary Motion.—In this chapter will be taken up what we will call complete rotary motion; that is, such motion as that of the gearing wheels of a machine, its fly-wheels, etc. With this motion any point of a wheel makes one complete circle after another in the same direction. Of course in this motion, as in any other motion about a fixed axis, the general formulæ for moment equation, energy equation, etc., are used; the reason for the special chapter being the practical importance of this sort of motion and the frequency with which it is encountered. A few practical examples should clear up the subject.

152. Uniform Rotary Motion.—A fly-wheel, or cylinder, which we will assume is of homogeneous material, when revolving at a *uniform* angular velocity about a smooth axis, is subjected to no resistance to rotation; for, the inertia of any particle acts radially *out* from the center of motion and is exactly balanced by the inertia of a particle which is diametrically opposite and at the same distance from the center of motion; also, the accelerating force, which acts *toward* the center of motion is balanced by that force acting on the diametrically opposite particle. The motion being *uniform* and everything *smooth* there are no other forces acting. This is the ideal condition, but is never possible practically.

With the practical fly-wheel, or cylinder, there are other forces to be considered; the friction on the axle, air resistance, etc., act-

ing to retard the motion. Therefore, to keep the wheel moving at a uniform speed, force must be applied in the direction of its motion to counteract the effect of the retarding forces, and the moment of the retarding forces, about the center of motion, must be equal to the moment of the counteracting force about the center of motion. This counteracting force is usually applied tangentially by means of a belt and we will call it the "belt force."

The friction is a tangential force which acts on the axle surface, and its moment is the product of the friction and the axle radius. The belt force may be calculated and its moment is the force times the belt radius. The moment of the air resistance is the difference between the moments of the belt force and the friction (assuming there are no other forces, such as useful work resistance).

Obviously, for a wheel to continue to rotate at a *uniform* speed, the moments of all the forces resisting rotation must be balanced by the moment of some force which encourages rotation. As the force which encourages rotation may be called the belt force, we may write as an equation

$$\left. \begin{array}{l} \text{Moment of} \\ \text{belt force} \end{array} \right\} = \left\{ \begin{array}{l} \text{Sum of the moments of} \\ \text{resisting forces.} \end{array} \right. \quad (1)$$

153. Accelerated Rotary Motion.—The conditions of the wheel in the preceding article were those of any wheel in a machine *after* it has attained the uniform speed at which the machine is worked. In attaining that speed all the parts are *accelerated* from rest and when the machine is stopped, all the parts are *retarded* to rest again. The conditions while acceleration or retardation are going on are different from those of uniform motion. During *acceleration*, the moment of the *belt force must exceed* the sum of the moments of the retarding forces, and during *retardation*, the sum of the moments of the *retarding forces must exceed* the moment of the belt force. In the latter case, retardation, the positive belt force is usually removed altogether, and the belt, etc., which must

be moved at the expense of some of the kinetic energy of the moving parts, acts as a drag and increases the amount of the ordinary retarding forces.

We have seen in the preceding article that the moments of forces acting during *uniform* rotation balance each other; we may then, when a change of angular velocity is occurring, that is, during acceleration or retardation, investigate the action of the *excess* of moment which causes the change of angular velocity; that is, we may ignore those parts of the moments which balance each other and concern ourselves only with the algebraic sum of the moments of *all* the impressed forces, this moment being a measure of the effect of the excess of external force.

Consider a particle, at a distance r from the center of motion, in any rotating body which is being accelerated. It has, as the angular velocity is changing, a tangential or linear acceleration. The particle diametrically opposite also has a tangential acceleration, but in the *opposite direction* to that of the first particle, so that the two tangential accelerations do not balance, though they are equal in magnitude, because the one is on the opposite side of the center of motion to the other and their effects must therefore be *added* together. These conditions are the same for every pair of particles in the rotating body.

Using the ordinary notation, the linear acceleration of each particle is,

$$a_r = \frac{dV_r}{dt} = \frac{d^2s}{dt^2} = r \frac{d\tilde{\omega}}{dt} = r \frac{d^2\theta}{dt^2} = r\beta. \quad (1)$$

The inertia of each particle is equal to ma , and the moment of the inertia force is rma , or,

$$rma_r = rm \frac{dV_r}{dt} = rm \frac{d^2s}{dt^2} = r^2m \frac{d\tilde{\omega}}{dt} = r^2m \frac{d^2\theta}{dt^2} = r^2m\beta. \quad (2)$$

The sum of the moments of the inertia force on *all* the particles, which is the measure of the effort which *resists* a change of angular velocity, must be equal to the moment of the force which *causes* the change of angular velocity; therefore, the equation of motion, noting that $\Sigma r^2 m$ is the moment of inertia of the whole mass, is

$$\left. \begin{array}{l} \text{Moment of the} \\ \text{accelerating force} \end{array} \right\} = \Sigma r^2 m \frac{d^2 \theta}{dt^2} = I \beta = I \frac{d^2 \theta}{dt^2}. \quad (3)$$

Assuming a smooth axle, no air resistance, and a belt force equal to F , the belt radius being b ; or, if the *excess* moment of the external (impressed) forces is bF , equation (3) becomes,

$$bF = I \beta = I \frac{d^2 \theta}{dt^2}. \quad (4)$$

154. Energy Equation of Rotary Motion.—We have seen (Art. 145) that the kinetic energy, at any instant, of bodies rotating about a fixed axis, is $\frac{1}{2} I \bar{\omega}^2$; therefore, any body rotating about a fixed axis with an angular velocity $\bar{\omega}$, can do work equal in amount to $\frac{1}{2} I \bar{\omega}^2$ before coming to rest. Let us suppose that a body, rotating about a fixed axis with a uniform angular velocity $\bar{\omega}$, requires a belt force F , acting at a distance b from the center of motion, to keep it moving at a uniform angular velocity; then Fb measures the sum of the moments of the retarding forces. Now suppose the belt force F , to be withdrawn, and suppose the body comes to rest after moving through the angle θ . Then the forces resisting the motion; *i. e.*, those retarding the body to rest, have the same moment as the force F acting at a distance b from the center of motion, and the body having turned through the angle θ , the relative space through which they must have acted, will have been $b\theta$; therefore, the work done against the resisting forces is equal to $Fb\theta$. This work must be equal in amount to the kinetic energy

that the revolving mass possessed at the instant the belt force was removed, or

$$Fb\theta = \frac{1}{2}I\tilde{\omega}^2. \quad (1)$$

From which any one unknown quantity may be obtained, when the conditions are those between *rest* and the angular velocity $\tilde{\omega}$.

For a general equation, we may eliminate t from the equation of motion, for $\tilde{\omega}$ is equal to $\frac{d\theta}{dt}$, or $dt = \frac{d\theta}{\tilde{\omega}}$, which, substituted in the equation of motion,

$$bF = I \frac{d^2\theta}{dt^2} = I \frac{d\tilde{\omega}}{dt}, \quad (2)$$

gives

$$bF \cdot d\theta = I\tilde{\omega}d\tilde{\omega}. \quad (3)$$

Integrating (3) between the limits $\tilde{\omega}_1$ and $\tilde{\omega}_2$ for $\tilde{\omega}$ and θ_1 and θ_2 for θ , we have

$$\frac{1}{2}I(\tilde{\omega}_2^2 - \tilde{\omega}_1^2) = bF(\theta_2 - \theta_1). \quad (4)$$

Equation (4) shows that the change in kinetic energy is equal to the work done while the mass rotates through the angle $(\theta_2 - \theta_1)$.

155. Illustrative Problems.—(a) The belt force, Art. 142, on a fly-wheel whose weight, 128 pounds, is considered to be uniformly distributed on the circumference of a circle of radius two feet, acts on a belt wheel one foot in radius. If the total amount of the belt force is 10 pounds, neglecting friction and air resistance, find the angular acceleration; the angular velocity and kinetic energy three seconds from rest; and the whole angle the wheel has turned through during the three seconds.

The mass of the wheel is $\frac{128}{g}$. The moment of inertia of the mass is $\frac{4 \times 128}{g}$, as all of the mass is supposed to be at a distance of two feet from the center of motion. The 10 pound belt force

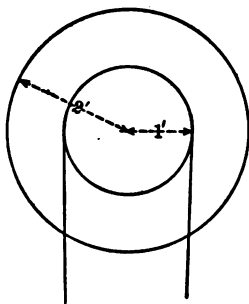


FIG. 67.

acts at a distance of one foot from the center of motion. The equation of motion is

$$bF = I\beta,$$

therefore,

$$10 = \frac{4 \times 128}{g} \cdot \frac{d^2\theta}{dt^2}. \quad (1)$$

From which

$$\beta = \frac{d^2\theta}{dt^2} = \frac{10g}{4 \times 128} = \text{ang. acc.} \quad (2)$$

Integrating (2) we have

$$\dot{\omega} = \frac{d\theta}{dt} = \left[\frac{10 \cdot g \cdot t}{4 \times 128} \right]_{t=0} = \frac{30g}{4 \times 128}. \quad (3)$$

Which gives the angular velocity three seconds from rest

$$\frac{1}{2}I\dot{\omega}^2 = \frac{4 \times 128}{2g} \cdot \left(\frac{30g}{4 \times 128} \right)^2 = \frac{225g}{2 \times 128}. \quad (4)$$

Integrating (3) we have

$$\theta = \left[\frac{10gt}{4 \times 128} = \frac{10gt^2}{8 \times 128} \right]_{t=0} = \frac{90g}{8 \times 128}. \quad (5)$$

Equation (5) gives the whole angle, in radians, turned through during the three seconds from rest.

(b) An engine exerts a mean turning moment of 1000 pounds-feet and the mean of all resisting forces is 300 pounds, which may be considered as acting at a distance of $2\frac{2}{3}$ feet from the center of motion. The radius of gyration of the fly-wheel mass is five feet, and the fly-wheel mass weighs one ton. Assuming the moving parts consist only of the fly-wheel, find the linear velocity at which a point, five feet from the center of the wheel, is moving at the end of a minute.

The resisting moment is $300 \times \frac{8}{3} = 800$ pounds-feet. The effective moment is $1000 - 800$ pounds-feet. The moment of

inertia of the mass is $\frac{25 \times 2240}{g}$. The equation of motion is, therefore,

$$200 = \frac{25 \times 2240}{g} \cdot \frac{d^2\theta}{dt^2}. \quad (1)$$

Integrating as in the previous problem,

$$\dot{\omega} = \frac{d\theta}{dt} = \frac{200 \times g \times 60}{25 \times 2240} = \frac{3g}{14}. \quad (2)$$

Therefore the required linear velocity of the point is $\frac{15g}{14}$, or about 34.5 f/s. *disk*

(c) A solid wheel, of weight W and radius a , turns on a smooth, weightless axle, of radius b , bolted to it. Wound on the axle is a fine cord to which is attached a weight W_1 . The weight W_1 is released, starting the wheel from rest, find the tension of the fine cord.

The motion of W_1 is linear and that of W is rotary. At rest the tension of the cord will be equal to the weight W_1 , but as soon as motion begins the inertia of W_1 will act *upward* and therefore reduce the tension of the cord. After the motion begins, we consider the two masses separately, and write two equations of motion; the one for the rectilinear motion of W_1 and the other for the rotary motion of W . Calling the tension of the cord T , the tangential force, acting to turn the wheel, will be T acting at a distance b from the center of motion. The force under which W_1 moves will be $W_1 - T$, acting downward. The moment of inertia of the mass of the solid wheel about its axis, which is perpendicular to its plane and through its center, is $\frac{a^2}{2} \cdot \frac{W}{g}$.

The two equations of motion are

$$W_1 - T = \frac{W_1}{g} a = \frac{W_1}{g} \cdot \frac{d^2s}{dt^2}, \quad (1)$$

and

$$bT = \frac{a^2}{2} \cdot \frac{W}{g} \cdot \frac{d^2\theta}{dt^2}. \quad (2)$$

The cord being inextensible, we have $\frac{d^2s}{dt^2} = b \cdot \frac{d^2\theta}{dt^2}$, which substituted in (1) gives

$$\frac{d^2\theta}{dt^2} = \frac{(W_1 - T)g}{W_1 b}. \quad (3)$$

Putting this value of $\frac{d^2\theta}{dt^2}$ in (2), we have

$$bT = \frac{a^2}{2} \cdot \frac{W}{g} \cdot \frac{(W_1 - T)g}{W_1 b}. \quad (4)$$

Solving (4) for T gives

$$T = \frac{a^2 W W_1}{2 W_1 b^2 + a^2 W}. \quad (5)$$

Problems XIX.

(Use $g = 32$ and $\pi = 2\frac{2}{7}$.)

1. A cylinder, 1 ft. in diameter, weighs 100 lbs., and has a fine cord wound on it to the end of which is attached a weight of 20 lbs. The cylinder is horizontal and free to revolve about its axis, what are the acceleration, tension of the cord, and kinetic energy, 5 seconds from rest?

2. A fly-wheel (weight in rim) weighs 10,000 lbs. and is 24 ft. in diameter. It turns against 400 lbs. of friction, acting on its axle, which is a foot in diameter. The wheel is making 15 turns a minute when steam is turned off, how many turns will it make before being at rest?

3. A wheel, 6 ft. in diameter and making 50 revolutions per minute, has a moment of inertia of mass equal to 600 in lbs.-ft. units. What tangential force applied to its rim for one minute will bring it to rest?

4. A grindstone, 2 ft. in diameter and weighing 128 lbs., is making 3 revolutions per second. What tangential force applied to its rim will bring it to rest in 6 seconds? What is its original energy of rotation?

5. A fly-wheel, weight 400 lbs., mean radius 3 ft., is turned by a weight of 50 lbs. which hangs from a cord wound on the axle, radius $\frac{1}{2}$ ft. Find the acceleration and angular velocity 3 seconds from rest.

6. A solid disc weighs 2000 lbs. Its radius of gyration is 3.2 ft., and it is making 60 revolutions per minute. It is carried on a shaft which is 3 in. in diameter, and which is turning against a uniform friction of 50 lbs. In what time will the disc come to rest and how many revolutions will it make before doing so?

7. Find the kinetic energy of a top, which is a right cone in shape, 1 in. radius of base and 2 in. in altitude, when it is making 4 revolutions per second. Its weight is half a pound.

8. A solid steel sphere, 2 ft. in diameter, is making 40 turns a second about a frictionless axis through its center. What is the least force, which, if applied tangentially to its surface for one minute, will stop it? Steel weighs 480 lbs. per cu. ft.

9. A mass has 20,000 ft.-lbs. of work done on it to increase its speed from 68 to 70 revolutions per minute. What is its moment of inertia of mass?

10. An axle, diameter 1 ft. and weight 64 lbs., is bolted to a solid wheel, diameter 6 ft., weight 128 lbs. A weight of 4 lbs. hangs from a cord wound on the axle and a weight of 2 lbs. hangs from a cord wound on the wheel. These weights hang down on opposite sides of the center of the wheel. Find the number of turns the wheel is making after 4 minutes and 55 seconds from rest.

CHAPTER XX.

KINETIC EQUILIBRIUM.

COMBINED ROTATION AND TRANSLATION.

156. Describes Combined Rotation and Translation.—A body is subject to a *motion of translation* when the paths of every particle composing the body are equal and parallel. In a body, subject to translation only, each particle, at any instant, has the same velocity (which includes both magnitude and direction); therefore, the motion of the body is determined if we know the condition of motion of any one particle. It is usual to select the particle at the center of gravity to represent the motion of the body.

A body is subject to a *motion of rotation* when the path of each and every particle in the body is a circle, the centers of all the circular paths being on a fixed straight line which is called the *axis of rotation*. In a rotating body each particle, at any instant, has the same *angular velocity* about the axis of rotation; therefore, the motion of a single particle will determine the motion of the body.

When a body is subjected to a combination of the two motions above, the paths of no two particles can be either equal parallel lines or circles, though were the body freed from its motion of rotation it would be subject to simple translation only, and vice versa.

As an illustration of combined rotation and translation, suppose a right cylinder is rolling on a horizontal plane. The whole mass is subject to a motion of translation along the plane, the space

traversed being the distance along the plane that the center of gravity has moved. During all the time the mass of the cylinder has been moving along the plane, the mass also has been turning about the cylinder's axis, and had there been no motion of translation, this latter motion would have been simple rotation; but, the axis *has* been moving (it has the motion of translation only) so we say the cylinder has a motion of rotation *relative* to its axis.

Looking at this combined motion in *another* way; at *any instant* any particle of the mass of the cylinder has an angular motion (velocity, acceleration) about the line of contact of the cylinder with the plane on which it rolls. The line of contact is called the *instantaneous axis of rotation*, and any particle, at all times, is moving at right angles to the perpendicular from the particle to the instantaneous axis of rotation. Viewing the motion in this way, the linear velocity of a particle at any instant may be resolved into two components, one of which will represent the velocity of translation and the other, the linear velocity of the particle, perpendicular to the line connecting the particle to the *axis of the cylinder*; in other words, the latter (tangential) component is the velocity, in a circle, *relative* to the axis of the cylinder, and the horizontal component is the velocity of translation of the whole cylinder.

It should be noted that the tangential component (which is velocity *relative* to the cylinder's axis) is a maximum on the curved surface of the cylinder and decreases, for a point within the cylinder, with the distance of the point from the axis of the cylinder, being zero at the axis; also, that the horizontal component is constant and equal to the velocity of translation at all points. This is as it should be when we remember that we have resolved the linear velocity, with reference to the *instantaneous axis* or line of contact, into components which are always *horizontal* and *tangential* to a circle whose center is on the axis of the cylinder.

157. Illustrative Problem.—Let Fig. 68 represent the section of a cylinder, of radius a , which is rolling at a uniform velocity, v , along a horizontal plane. Then, if the angular velocity with which the cylinder rotates about its axis is ω , we have $v = a\omega$. Let C be the center of the circular section, θ the angle through which the cylinder has rolled since the instant our observations

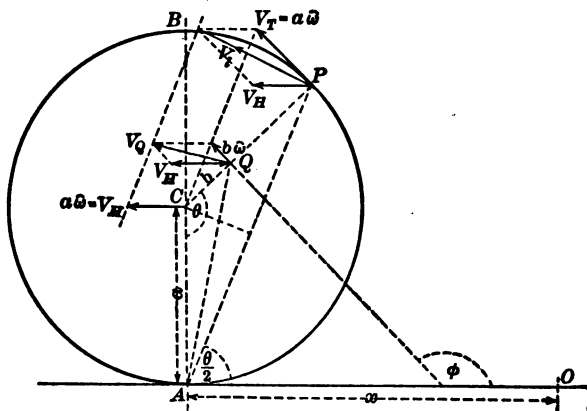


FIG. 68.

began, and x the distance it has moved along the plane in the time t which has elapsed; then, $x = a\theta$, or

$$\frac{x}{t} = a \frac{\theta}{t} = a\omega = v. \quad (1)$$

Let P be the point on the curved surface of the cylinder which was in contact with O when our observation began, then its distance from the instantaneous center of rotation, A , is equal to $2a \sin \frac{\theta}{2}$. The angle PAO is equal to $\frac{\theta}{2}$, it being an angle at the circumference measured by the same arc as θ ; also $\theta = \omega t$; there-

fore, the linear velocity of the point P , which is in a direction at right angles with PA , is

$$V_i = 2a \sin \frac{\theta}{2} \cdot \frac{\theta}{t} = 2a\omega \cdot \sin \frac{\theta}{2}. \quad (2)$$

This linear velocity may be resolved into components which are horizontal and tangential to the cylinder at P . Now, *relative to* C , the point P describes a circle of radius a , hence the linear velocity, at any instant, relative to C , is $a\omega$, and it is directed along the tangent at P as indicated. This, then, must be the tangential component of V_i , and the horizontal resolved part of this tangential component is $a\omega \sin \left(\theta - \frac{\pi}{2} \right) = -a\omega \cos \theta$. Now the horizontal resolved part of V_i is $2a\omega \cdot \sin \frac{\theta}{2} \cdot \cos \left(\frac{\pi}{2} - \frac{\theta}{2} \right)$, and this horizontal resolved part includes the horizontal part of the tangential component of V_i ; therefore, that resolved part, $-a\omega \cos \theta$, must be subtracted from the horizontal resolved part of V_i which will leave the horizontal component of the linear velocity V_i , or

$$2a\omega \sin^2 \frac{\theta}{2} + a\omega \cos \theta = a\omega (1 - \cos \theta) + a\omega \cos \theta = a\omega, \quad (3)$$

which is, as it should be, the velocity of translation. Referring to the figure, the result of (3) may be derived directly by solving the following equation for V_H ,

$$V_i^2 = V_H^2 + V_T^2 + 2V_H V_T \cos (\pi - \theta). \quad (4)$$

From the figure, noting that the tangential component of the total linear velocity of a point Q , distant b from C and on the radius PC , is $b\omega$, we see that the tangential component of the total linear velocity varies directly with the distance from C , while the horizontal component remains constant, becoming the total linear velocity when considering the point C itself, where the tangential component is zero. Again, the total linear velocity at the upper

end, B , of the vertical diameter is $2a\tilde{\omega}$, all of which is horizontal, but half of which is the velocity of translation and the other half the tangential velocity due to the motion of rotation relative to C ; at the lower end, A , of the vertical diameter the total linear velocity is zero, and it is the *algebraic* sum of the velocity of translation and the tangential velocity due to the motion of rotation relative to C , both these velocities being again horizontal at A , but the tangential velocity due to the relative rotation now acts to the right.

158. Kinetic Energy of Combined Translation and Rotation.—

Let us use the cylinder of the preceding article and refer to Fig. 68. The axis of the cylinder is the axis of relative rotation and we have seen that for a point on the curved surface of the cylinder the linear velocity of rotation is $a\tilde{\omega}$, and that this velocity is equal to the velocity of translation, v , or $v = a\tilde{\omega}$, from which the angular velocity is $\frac{v}{a}$ radians per second. A mass m at Q , distant r from C (b of the figure being replaced by r , thus making the case general), has the horizontal velocity of translation $a\tilde{\omega}$, and also the velocity of rotation, relative to C , of $r\tilde{\omega}$, this latter being in the direction of the tangent to the circle described by Q , and thus making the angle, $\phi = \theta$, with the direction of the motion of translation. The angle between the two velocities possessed by Q will be $(\pi - \theta)$ and the resultant of these two velocities will be the total linear velocity possessed by the mass at Q . This resultant is

$$V^2_Q = a^2\tilde{\omega}^2 + r^2\tilde{\omega}^2 + 2a\tilde{\omega} \cdot r\tilde{\omega} \cdot \cos(\pi - \theta),$$

or

$$V^2_Q = a^2\tilde{\omega}^2 + r^2\tilde{\omega}^2 - 2a\tilde{\omega} \cdot r\tilde{\omega} \cdot \cos \theta. \quad (1)$$

The kinetic energy of the mass, m , is $\frac{1}{2}mV^2_Q$, and for the whole cylinder,

$$\text{K. E.} = \Sigma \frac{1}{2}m(a^2\tilde{\omega}^2 + r^2\tilde{\omega}^2 - 2a\tilde{\omega} \cdot r\tilde{\omega} \cdot \cos \theta),$$

or

$$\text{K. E.} = \frac{v^2}{2} \Sigma m + \frac{1}{2} \tilde{\omega}^2 \Sigma m r^2 - \tilde{\omega} v \Sigma m r \cos \theta. \quad (2)$$

But $\Sigma m r \cdot \cos \theta$ is equal to zero for a cylinder, and $\Sigma m r^2 = I$, the moment of inertia of the mass of the cylinder about its geometric axis; therefore,

$$\text{K. E.} = \frac{1}{2} M v^2 + \frac{1}{2} I \tilde{\omega}^2. \quad (3)$$

As $I = k^2 M$ and $\tilde{\omega}^2 = \frac{v^2}{a^2}$, equation (3) may be written

$$\text{K. E.} = \frac{1}{2} M v^2 \left(1 + \frac{k^2}{a^2} \right) = \frac{M v^2}{2} \left(\frac{a^2 + k^2}{a^2} \right). \quad (4)$$

Equation (3) shows, what we might have taken for granted, that the total K. E. of a body subjected to the combined motions of translation and rotation is the sum of the kinetic energy of translation and the kinetic energy of relative rotation about the body's axis.

159. Accelerated Motion of Combined Rotation and Translation.—Before proceeding with the accelerated motion of combined rotation and translation, it should be noted that in the preceding articles the bodies are assumed to be *rolling* along a plane with a uniform speed such that the motion of translation is connected with the relative motion of rotation by the equation, $v = a\tilde{\omega}$. Now on a smooth plane there is no reason why a body should roll unless it is initially *given* a rolling motion; for, if given a motion of translation only, the body would *slide* along the plane as the plane is perfectly smooth; therefore, to obtain rolling motion such as we have been considering, without, *at the start*, giving the body the required motion of rotation, the plane must be rough enough so that the friction, at the line of contact between the body and the plane, will be sufficient to prevent sliding. Assuming this condition from now on; a case of accelerated motion of combined rotation and translation is that of a body

rolling down a rough plane, inclined at an angle α with the horizontal. Supposing the body a cylinder, of radius a , and that it starts from rest, we know that the kinetic energy it has, at any instant, is equal to the work done by gravity up to that instant; therefore, using the results of the preceding article, and remembering that the work done by gravity, is the force, $W \sin \alpha$, multiplied by the space, s , traversed along the plane, we have

$$\frac{1}{2} M v^2 \left(1 + \frac{k^2}{a^2} \right) = W \sin \alpha \cdot s, \quad (1)$$

from which

$$v^2 = 2 \cdot g \sin \alpha \left(\frac{a^2}{a^2 + k^2} \right) \cdot s. \quad (2)$$

Comparing (2) with the equation, $v^2 = 2 \cdot g \sin \alpha \cdot s$, giving the squared velocity acquired in *sliding* a distance s down an inclined plane, we see that the acceleration, for the rolling body, is

$$\text{Acceleration} = g \sin \alpha \left(\frac{a^2}{a^2 + k^2} \right). \quad (3)$$

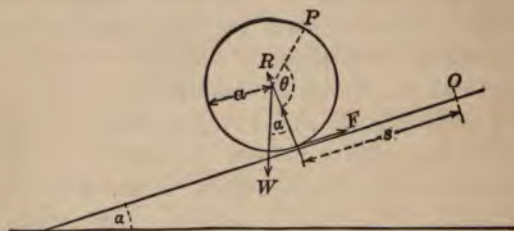


FIG. 69.

It may be interesting to obtain this result by a general method going back to Newton's formula that the force is equal to the product of the mass and the acceleration. Referring to Fig. 69, let P be the line (perpendicular to the paper) of contact between the cylinder and the plane, when the cylinder was at rest, and let O be the corresponding line on the inclined plane; then $s = a\theta$.

The forces which act on the cylinder are, its weight, W , the normal resistance, R , of the plane and the friction, F .

Resolving along and perpendicular to the inclined plane, the force which causes the motion of translation, is $W \sin \alpha - F$, acting along the plane, the resistance R and the normal component of W , being normal to the plane, have no effect on the motion of translation of the cylinder along the plane. The equation of the motion of translation will be, therefore,

$$m \frac{d^2 s}{dt^2} = W \sin \alpha - F. \quad (4)$$

Now the cause of the rotation of the cylinder is the tangential frictional force F , which acts up the plane; therefore, relative to C , the equation of rotary motion is

$$aF = I \frac{d^2 \theta}{dt^2} = k^2 m \frac{d^2 \theta}{dt^2}. \quad (5)$$

Multiplying (4) by a and substituting for $\frac{d^2 s}{dt^2} = a \frac{d^2 \theta}{dt^2}$, we have

$$Fa = Wa \sin \alpha - ma^2 \frac{d^2 \theta}{dt^2}. \quad (6)$$

Putting this value of Fa in (5) gives

$$Wa \sin \alpha - ma^2 \frac{d^2 \theta}{dt^2} = k^2 m \frac{d^2 \theta}{dt^2}. \quad (7)$$

Whence

$$\frac{d^2 \theta}{dt^2} = \frac{Wa \sin \alpha}{m(k^2 + a^2)} = \frac{ga \sin \alpha}{k^2 + a^2}, \quad (8)$$

or

$$\frac{d^2 s}{dt^2} = \frac{a^2}{a^2 + k^2} \cdot g \sin \alpha. \quad (9)$$

which agrees with (3).

The squared radius of gyration of a cylinder about its geometric axis is $\frac{a^2}{2}$; therefore, the acceleration of translation, down the incline, is $\frac{2}{3} \cdot g \sin \alpha$. Comparing this with the acceleration of a body *sliding* down a *smooth* inclined plane, we see that *rolling* reduces the linear acceleration and velocity; in fact, for this cylinder, one-third of the natural acceleration (the retardation due to friction is insignificant) is expended in causing the body to rotate on its axis.

Integrating (9), noting that the constant of integration is zero as the cylinder starts from rest, we have

$$v = \frac{ds}{dt} = \frac{a^2}{k^2 + a^2} \cdot g \cdot \sin \alpha \cdot t, \quad (10)$$

and

$$s = \frac{1}{2} \cdot \frac{a^2}{k^2 + a^2} \cdot g \cdot \sin \alpha \cdot t^2. \quad (11)$$

for the velocity of translation and the space traversed in time t .

Integrating (8) will give the angular velocity and the whole angle, about its axis, that the cylinder has turned through in the time t .

160. Screw Motion.—The motion of a screw which works in a fixed nut, or of a nut which moves on a fixed screw thread, are examples of combined translation and rotation. In the motion of the screw in the fixed nut, the translation takes place along the axis of the screw and the rotation about the axis of the screw. The whole kinetic energy is expressed by equation (3) of Art. 158, being the sum of the kinetic energies of translation and of rotation. In the case of the nut working on a fixed screw thread, the conditions are the same, and the above equation expresses the value of the total kinetic energy; care must be taken, however, to use for I the moment of the inertia of the mass of the nut about the *axis of the screw*.

Problems XX.

1. A cylinder weighing 20 lbs., is allowed to roll down an incline of 30° . Through what space will it move in 10 seconds? What will be its velocity after 10 seconds and what is then its kinetic energy?

2. What is the velocity of the cylinder of problem 1, after it has traversed 12 ft. down the inclined plane?

3. A solid sphere, weighing 20 lbs., is allowed to roll down an inclination of 30° . Find its acceleration and through what space it will move down the plane in 10 seconds from rest.

4. The radius of the sphere of problem 3 is 1 ft., find the kinetic energy of translation and of rotation after 10 seconds from rest.

5. Find the acceleration of translation of a spherical shell which is allowed to roll down an inclination of α with the horizontal. *mom. of in. of surface of shell = $\frac{2}{3}m\alpha^2/L$*

6. Find the acceleration of translation of a hoop which is allowed to roll down an inclination of α with the horizontal.

7. A cylinder has a fine cord wound on its surface. The end of the cord is fast to a fixed point, and the cylinder is let fall. Assuming that the cylinder's axis remains horizontal, find the acceleration of translation.

8. A wagon, which weighs 2 tons, has wheels 2 ft. in diameter. The wheels weigh $\frac{1}{4}$ of a ton and their radius of gyration is .9 ft. What is the kinetic energy of the wagon when moving at 40 m/h?

9. Show that in problem 1, the cylinder will slip some as it goes down the slope, unless μ , the coefficient of friction, is less than $\sqrt{3}$.

10. Suppose a perfectly free rod of length b is struck a quick blow at a distance a from its center of gravity and in a direction normal to the rod. Discuss the motion produced.

Chapt. XX_a

Angular Momentum

Sl. 408 - 424

APPENDIX.

CENTER OF GRAVITY.

The center of gravity of any mass is the *point of application* of the earth's attractive force for that mass.

The earth attracts every particle of matter toward the earth's center. The lines of action, of the force of this attraction on different particles, therefore converge toward the earth's center, but when we consider the length of the earth's radius as compared with the distance between particles, the lines of action are practically parallel within considerable limits of area on the earth's surface. We may find, therefore, in the usual way, the resultant of the system of *parallel* forces with which the earth attracts all the particles of matter of any body, and this resultant will be the earth's total attractive force for that body. The line of action of this resultant will obviously be vertical and the *point of application* must be somewhere on it. If we turn the mass through an angle of 90° in the vertical plane and again find the line of action of the resultant earth's attractive force the new line of action must intersect the old one in the *point of application*, for this point must lie on both lines.

Now, as the earth's attractive force on one particle of matter of any body is the same as it is for any other particle of matter of that body, the problem of fixing the position of the point of application of the earth's attractive force on a body resolves itself into finding the mean distances, of all the particles of matter a body possesses, from three planes which intersect each other at right angles.

We have from the Calculus (the distance of a particle dm from the plane YZ being x) the *sum of the products*, $x dm$ (the mo-

ment), for all the particles in a body, *divided by* the number of the particles, will be the mean distance of all from the YZ plane, or

$$\bar{x} = \frac{\int x dm}{\int dm}. \quad (1)$$

Mean distances from the XZ and XY planes are found, in the same way, to be

$$\bar{y} = \frac{\int y dm}{\int dm}, \quad \text{and} \quad \bar{z} = \frac{\int z dm}{\int dm}, \quad (2)$$

where $\bar{x}$, $\bar{y}$ and $\bar{z}$ are the co-ordinates of the center of gravity.

Center of gravity may be defined as the point at which the whole weight of a body acts and thus, if a body could be supported at its c. g., it would be in equilibrium in whatever position it might be put.

Again, the position of the c. g. of many bodies is known from the symmetry of their figures; as, the c. g. of a sphere is at the center of the sphere; that of a cube is at its center (intersection of the diagonals), etc.

If we suppose a line to be a single row of particles of matter and a surface to be a lamina of a uniform thickness of one particle of matter, then we may find the c. g. of lines and surfaces. From symmetry the c. g. of a straight line is at its middle point (supposing the particles to be equally spaced as they are in homogeneous masses) and the c. g. of a *plane* surface must lie *in the surface*; in which cases, for *the straight line*, we have only to find the mean distance of all the particles from *one* plane, and for *the plane surface*, the mean distance of all the particles from *two* planes. We are able, in many cases (rectangles, circles, etc.), to fix the position of the c. g. of plane surfaces from symmetry; also, in a few cases of curved surfaces such as cylindrical and spherical surfaces.

With ordinary materials, wood, stone, iron, etc., we must take it that the particles of matter are uniformly spaced (of course this

is not quite true because of the knots in the wood, denser spots in the stone, blow holes, etc., in the iron); but sometimes things are made (constructed) to be of a *variable* density, in which case the density must be taken into account. This may be done by substituting in the formula, ρdv for dm , but the law of the variation of the density must be known.

When the density is uniform, the elements of area dA , of arc ds , and of volume dv , may be used instead of the element of mass in the above formulæ.

Again, since the moment of the *whole mass* must equal the sum

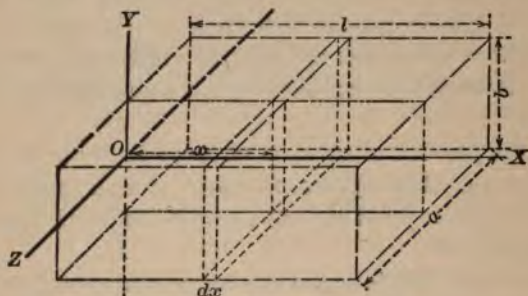


FIG. 70.

of the moments of *its parts*, we may divide any solid or surface into finite parts and form an equation by putting the sum of the moments of the parts about any line, equal to the moment of the whole about the *same* line. The moment of one of the parts is the product of the mass of that part and the perpendicular distance of the c. g. of the part from the axis of moments. From each equation of this sort one unknown quantity may be obtained. Thus knowing the position of the c. g. of the parts, we may find the co-ordinates of the c. g. of the whole by taking moments about the three co-ordinate axes.

Let us find the c. g. of a uniform straight line of length l . Taking the origin at one end, the moment of any portion, $dm = dx$,

of the line, at a distance x from the end, is $x \cdot dx$. The integral of this is the sum of the moments of all the elements of mass. Dividing this sum by the sum of all the elements of mass, $\int dx$, we have

$$\bar{x} = \frac{\int_0^l x dx}{\int_0^l dx} = \frac{\left[\frac{x^2}{2} \right]_0^l}{\left[x \right]_0^l} = \frac{l}{2}. \quad (3)$$

Now let us suppose the line to have been of a uniform width b . The element of mass now becomes a small area of magnitude $b dx$, with which we replace dx in the above. The result is again $\frac{l}{2}$ and therefore the c. g. of a uniform rectangle is one-half its length from one end.

Again, let the above line be b wide and a thick. The element of mass is now a small volume, equal to $b a dx$. Replacing dm by this, we again find $\bar{x}$ equal to $\frac{l}{2}$; and therefore the c. g. of a uniform prism is one-half the length from one end.

Suppose, in the case of the prism, that the density increases directly with the distance from the origin (one end). Mass is equal to the product of the density and the volume; therefore, if the density at unit distance from the origin be ρ , the density at a distance x is ρx , and the mass at a distance x is $\rho x b a \cdot dx$. The moment of this mass is $\rho x^2 b a dx$, and the sum of the moments of all the elements of mass is $\int_0^l \rho x^2 b a dx$. The sum of all the elements of mass is $\int_0^l \rho x b a dx$, and the c. g. of a rod whose density varies directly with the distance from one end is

$$\bar{x} = \frac{\int_0^l \rho x^2 b a dx}{\int_0^l \rho x b a dx} = \frac{\left[\frac{\rho x^3}{3} \right]_0^l}{\left[\frac{\rho x^2}{2} \right]_0^l} = \frac{2}{3} l, \quad (4)$$

from the less dense end.

Very few natural things are of variable density so that in actual practice we have to do with things of uniform density; that is, with masses which are homogeneous.

Let us find the c. g. of the area of a semicircle of radius a . The equation of the curve, with origin at the center of the arc, is $x^2 + y^2 = a^2$. From symmetry, the c. g. must lie on the radius which is perpendicular to the bounding diameter, because there is an equal mass on either side of that radius. Therefore, assuming this radius as the axis of X , the bounding diameter as the axis of Y and using the point element, $dx dy$, we have

$$\begin{aligned}\bar{x} &= \frac{\int_0^a \int_0^{\sqrt{a^2-x^2}} x dx dy}{\int_0^a \int_0^{\sqrt{a^2-x^2}} dx dy} = \frac{\int_0^a x \sqrt{a^2-x^2} dx}{\int_0^a \sqrt{a^2-x^2} dx} \\ &= \frac{-\frac{(a^2-x^2)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^a}{\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \Big|_0^a} = \frac{\frac{a^3}{3}}{\frac{a^2\pi}{4}} = \frac{4a}{3\pi}. \quad (5)\end{aligned}$$

The c. g. of the arc is

$$\begin{aligned}\bar{x} &= \frac{\int x ds}{\int ds} = \frac{\int_0^a x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx}{\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx} \\ &= \frac{\int_0^a \frac{ax dx}{\sqrt{a^2-x^2}}}{\int_0^a \frac{adx}{\sqrt{a^2-x^2}}} = \frac{-\sqrt{a^2-x^2} \Big|_0^a}{\sin^{-1} \frac{x}{a} \Big|_0^a} = \frac{2a}{\pi}. \quad (6)\end{aligned}$$

Parametric and polar equations are frequently convenient; as,

in the case of the semicircular area above, using the parametric equations $x = a \cos \theta$ and $y = a \sin \theta$, from which $r = \sqrt{x^2 + y^2} = a$, we have the area of a triangular element equal to $\frac{a^2}{2} d\theta$, its c. g. being $\frac{2}{3}a$ from the center or $\frac{2}{3}a \cos \theta$ from the axis of Y (the bounding diameter). The moment of the triangular element about the axis of Y is $\frac{2}{3}a \cos \theta \cdot \frac{a^2}{2} d\theta$, and therefore

$$\bar{x} = \frac{\frac{a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta}{\frac{a^2}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta} = \frac{\frac{2a^3}{3}}{\frac{a^2\pi}{2}} = \frac{4a}{3\pi}. \quad (\gamma)$$

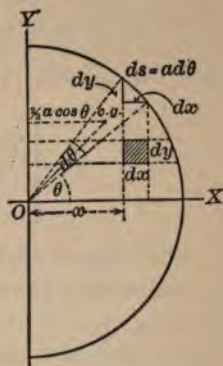


FIG. 71.

as above.

With polar equations, the area may be divided up into elementary triangles, as above; or, we may use the method of double integration, with which the point element, $r d\theta dr$, is required.

From symmetry, the c. g. of a solid of revolution must always lie on the axis of revolution.

Usually the most convenient element to use when finding the c. g. of a solid of revolution is a circular lamina perpendicular to the axis of revolution. For example, to find the c. g. of a right cone, assume the origin at the apex, and the axis of X to be the axis of revolution, then the axes of Y and Z will be in a plane parallel to the base of the cone and we may use

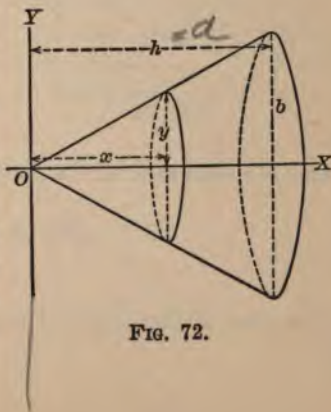


FIG. 72.

the equation derived from the proportion, the radius of the base is to the altitude as y is to x ; or, $y = \frac{b}{h} x$. The radius of the base and the altitude of the cone being, respectively, b and h . Then

$$\bar{x} = \frac{\int x \pi y^2 dx}{\int \pi y^2 dx} = \frac{\frac{\pi b^2}{h^2} \int_0^a x^3 dx}{\frac{\pi b^2}{h^2} \int_0^a x^2 dx} = \frac{\frac{x^4}{4} \Big|_0^a}{\frac{x^3}{3} \Big|_0^a} = \frac{3a}{4}. \quad (8)$$

With masses which are not solids of revolution, if the equation of the surface is known, the c. g. may be found either by triple integration or by the method indicated below. When the equation of the surface is not known the position of the c. g. may be determined by experiment. For the first case (equation known) let

us determine the position of the c. g. of that part of the ellipsoid, given by equation (9) for which the value of x is positive (i. e., half the solid).

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (9)$$

The equation of any section of this solid, parallel to the YZ plane and at a distance x from it, is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{a^2 - x^2}{a^2}, \quad (10)$$

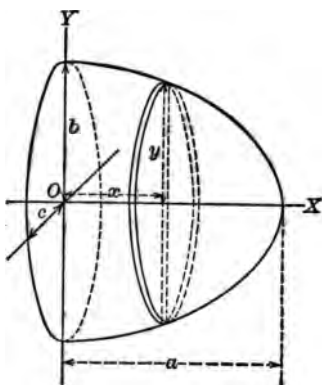


FIG. 73.

and the semi-axes of the section are

$$\frac{b}{a} \sqrt{a^2 - x^2} \quad \text{and} \quad \frac{c}{a} \sqrt{a^2 - x^2}.$$

The area of the elliptical section is $\frac{\pi b c}{a^3} (a^2 - x^2)$ and the ele-

mentary volume is $\frac{\pi bc}{a^2} (a^2 - x^2) dx$. The c. g. is at the center of this elementary volume and at a distance x from the YZ plane; therefore,

$$\bar{x} = \frac{\frac{\pi bc}{a^2} \int_0^a x(a^2 - x^2) dx}{\frac{\pi bc}{a^2} \int_0^a (a^2 - x^2) dx} = \frac{\frac{1}{4}\pi a^2 bc}{\frac{2}{3}\pi abc} = \frac{3}{8} a. \quad (11)$$

It may be noted that when a , b , and c are all equal we have a hemisphere; therefore, the c. g. of a hemisphere is $\frac{3}{8}$ of the radius up from the base.

Frequently solids are of such irregular shapes that the equations of their surfaces are impossible or so complicated as to be useless. Under such circumstances we may resort to the methods of *suspension* or *balancing*. In the first, a body is suspended by a cord and the direction of the cord (the line of action of gravity) is noted. The cord is then attached to another point of the body, the body suspended and the direction of the cord noted. The intersection of the two directions of the cord is at the c. g. of the body.

The method by balancing is the reverse of the above. The body is balanced on a knife edge; the c. g. must lie in the vertical plane through the knife edge. After balancing in three different positions, the c. g. is at the point where the three planes intersect. With *flat* masses it is usually sufficient to balance but twice, the c. g. being half the thickness of the mass above the intersection of the two positions of the knife edge.

UNITS.

The units used throughout this book are those known as engineers' or gravitational units. Another system, arranged to suit the convenience of American and English scientists, is known as the system of *absolute units*. With the engineers' system, the fundamental units are those of *length*, *time* and *force*. From these, the other units are *derived* so that the fundamental formula, force equals mass times acceleration (Newton's law), remains true. With the absolute system, the fundamental units are those of *length*, *time* and *mass*. In this connection, fundamental units may be defined as units independent of other units for their value. For example, in the engineers' system, choice has been made of the foot, the pound, and the second, for the units of length, force and time. This made it necessary to *derive* the unit of mass so that Newton's law would hold good. We have seen in Chapter II how this was done, the unit of mass being that mass which weighs g pounds; or the mass of anything is numerically equal to $\frac{W}{g}$.

With the *absolute* system, the fundamental units chosen are *length*, *time* and *mass*; the units of length and time being the same (the foot and the second) as those in the engineers' system, but the unit of mass was chosen as the mass of one pound weight of anything. Thus, instead of the unit of mass being the mass of g pounds, as it is in the engineers' system, it is the mass of *one* pound. Thus the mass of anything is W (its weight) units in the absolute system. The *selection* of the unit of mass made it necessary to *derive* a unit of force, such that when unit force acted on unit mass it imparted unit acceleration. The absolute unit of *force* thus derived is called the *poundal* and it amounts to $\frac{1}{g}$ th of a pound, about half an ounce, or about $\frac{1}{16}$ of

the engineers' unit. The absolute system is used almost entirely by scientists and was chosen by them because with a *fixed mass unit* the *force unit* is independent of gravity, a condition desirable in scientific research.

The absolute unit of work is the foot-poundal; the amount of work done when one poundal acts through a space of one foot.

In the following table the units of the engineers' and absolute systems are compared:

	Engineers' Units.	Absolute Units.
Length	Foot.	Foot.
Time	Second.	Second.
Force	Pound.	Poundal = $\frac{1}{g}$ pounds.
Mass	Numerically equal to g pounds.	Numerically equal to the pound weight.
Work	Foot-pound.	Foot-poundal.
Energy	Foot-pound.	Foot-poundal.
Power	$\left\{ \begin{array}{l} \text{Ft.-lb. per sec.} \\ \text{HP} = 550 \text{ ft.-lbs.} \\ \text{per sec.} \end{array} \right.$	$\left\{ \begin{array}{l} \text{Foot-poundal per sec.} \\ \text{HP} = g \times 550 \text{ ft.-} \\ \text{pdl. per sec.} \end{array} \right.$
Velocity	Ft. per sec.	Ft. per sec.
Acceleration	Ft. per sec. per sec.	Ft. per sec. per sec.
Angular velocity..	Radian per sec.	Radian per sec.
Angular acceleration	Radian per sec. per sec.	Radian per sec. per sec.
Impulse	Pound-sec.	Poundal-sec.

Weight being a measure of the *force* of gravity, the unit of weight is the pound.

No name has been given the unit of mass as its numerical value is always expressed in terms of the weight of the mass.

Still other units, which we may term foreign units, are frequently employed (due to the convenience they offer in scientific calculations). These are the metric gravitational and the metric absolute systems.

In the *metric gravitational system*, the fundamental units are those of *length*, *time* and *force*. The unit of length being the centimeter, the unit of time the second, and the unit of force the gram. The acceleration of gravity in this system is 981 centimeters per second per second; or, $g=981 \text{ cm/s}^2$. In the same manner as was derived the unit of mass for engineers' units, we derive the numerical value of the unit of mass for the metric gravitational system of units. This numerical value, in terms of the gram, is 981. The gravity equation $W=mg$ is one gram equals $\frac{1}{981} \times 981$; that is, a body weighing one gram has $\frac{1}{981}$ units of mass, or the unit of mass must weigh 981 grams. This system is similar in all respects to the engineers' system except that the units are smaller. In fact these units are so small that engineers abroad use the kilogram (1000 grams) as the unit of force and the meter as the unit of length.

In the *metric absolute system*, the fundamental units are those of *length*, *time* and *mass*; the centimeter, the second and the mass of a body which weighs a gram, being, respectively, the chosen units. This system is, in all respects, similar to our absolute system. The unit of force is derived in the same manner and is called the *dyne*. It is equal to $\frac{1}{981}$ of a gram; or the weight of one gram (the force of gravity for a gram mass) is equal to 981 dynes.

In the following table the units of the metric gravitational and absolute systems are compared:

	Metric Gravitational.	Metric Absolute.
Length	Centimeter.	Centimeter.
Time	Second.	Second.
Force	Gram.	Dyne = $\frac{1}{981}$ grams.
Mass	{ Numerically equal to 981 grams.	{ Numerically equal to the gram weight.
Work		
	Cm. gram.	Cm. dyne, called <i>erg.</i>
Energy	Cm. gram.	Cm. dyne, called <i>erg.</i>
Power	Cm. gram per sec.	Cm. dyne per sec. (<i>erg/sec.</i>).
Impulse	Gram-sec.	Dyne-sec.

The velocities and accelerations are measured in both by centimeters per second and centimeters per second per second. The angular velocities and accelerations are measured in radians just as our own.

VIRTUAL WORK.

When the point of application of a constant force, F , suffers a displacement, s , along the line of action of the force, the work done amounts to Fs ; or, for an infinitesimal displacement, Fds . Now suppose the displacement, ds , does not actually take place, but is imagined to take place. It is then called a *virtual* displacement and the work, which would have been done had the displacement actually occurred, is called the *virtual work*. Virtual work, then, is imaginary work; that is, work which would have to be done if certain small displacements *should* occur. Virtual displacements are almost invariably assumed to be infinitesimal and consequently rectilinear. An *actual* displacement, which is *not* in the line of action of a force, would be due to that component of the force which is in the direction of the displacement, as we have seen; or, calling θ the angle between the line of action of the force, F , and the direction of the displacement, ds , the work done would be $F \cos \theta ds$, and *no* work is done if θ is 90° .

Now imagine a particle in equilibrium under the action of a number of forces, and assume it to suffer a *virtual* displacement in any direction whatever; the algebraic sum, of the separate works done by each force, must equal zero; for, the work done by each force is equal to that component of the force which lies in the direction of the virtual displacement multiplied by the displacement, and the algebraic sum of all these components of the forces must equal the component of the resultant in the direction of the virtual displacement. But the resultant is zero (forces in equilibrium), therefore the sum of the components of the forces must be zero and *no* work is done.

The conditions for the equilibrium of a particle are so simple that there is no special advantage in using the principle of virtual work when considering the action of force on a particle. In fact the work equations readily reduce to the equations for equilibrium.

But when we consider a rigid body as a system of particles, we know that *every* particle of the system must be in equilibrium if the system is in equilibrium. And, further, if a system of particles is *rigid*, the algebraic sum of the virtual works done by the *external* forces acting on the system must equal zero; for the *internal* forces in a rigid system all balance and the work done by them is obviously equal to zero.

The principle of virtual work is most useful in solving problems in statics, but may be employed for the problems of kinetics as well. Of course, in kinetics the particles may suffer actual displacements.

Suppose a heavy particle rests on a board, which is tilted to an angle θ with the horizontal, and is just on the point of sliding down (equilibrium). The forces acting on the particle are, its weight W , the friction F acting *up* the plane and the resistance R of the board acting normal to the board. Suppose a virtual displacement ds down the plane. The work done by W is $W \sin \theta ds$; by R is $R \cos 90^\circ ds = 0$; by F is $F ds$. Taking directions into account the sum of the works done by the three forces is $F ds - W \sin \theta ds$, which is equal to zero because the resultant is zero, the forces being in equilibrium. Putting the sum equal to zero we have directly $F = W \sin \theta$, which, as we know, is a condition for the equilibrium of the particle.

Of course, the selected direction, *down the plane*, for the virtual displacement is the natural direction for the motion to take. By choosing a virtual displacement down the plane, we have chosen a direction which is perpendicular to one of the forces (R , as yet unknown), thus eliminating that force from the equation for the virtual work. The equation therefore contains only *one* unknown quantity, F . The selection of some other direction for the virtual displacement might have given an equation containing *both* of the unknown quantities, R and F .

As an example of the principle of virtual work applied to a rigid body in equilibrium, let us take the simple case of the lever.

Referring to Fig. 74, let AB be the lever, C the fulcrum, P and Q the applied forces and R the force at the fulcrum. Choose any point O in the plane of the forces, and suppose the lever to turn about O through an infinitesimal angle $d\theta$, the forces remaining constant in magnitude and direction. The movements of particles at A , B and C , where the external forces act, are, respectively, $\overline{OA}d\theta$, $\overline{OB}d\theta$ and $\overline{OC}d\theta$. These motions being of different magnitudes as they depend on the distances OA , OB and OC and different directions as they are perpendicular respectively to the directions of OA , OB and OC . Let the angles between the directions of the forces P , Q and R and the directions of the virtual displacements of A , B and C , be, respectively, ϕ_1 , ϕ_2 and ϕ_3 (indicated in the figure). Then the components of the forces, P , Q and R , in the directions of the virtual displacements are, respectively, $P \cos \phi_1$, $Q \cos \phi_2$ and $R \cos \phi_3$. Noting *directions* of the works done, the sum, which is equal to zero as the lever is in equilibrium, of the works done by the three forces is

$$P \cos \phi_1 \overline{OA}d\theta - Q \cos \phi_2 \overline{OB}d\theta - R \cos \phi_3 \overline{OC}d\theta = 0. \quad (1)$$

If we drop perpendiculars from O on the lines of action of P , Q and R and call these perpendiculars p , q and r , respectively, we have

$$p = \overline{OA} \cos \phi_1; \quad q = \overline{OB} \cos \phi_2; \quad \text{and} \quad r = \overline{OC} \cos \phi_3. \quad (2)$$

Substituting the values of OA , OB and OC from (2) in (1), we have

$$Pp - Qq - Rr = 0. \quad (3)$$

From (3) the sum of the moments of the forces about any point in the plane of the forces is zero, as it should be, the lever being in equilibrium.

If we suppose the point C (the fulcrum) to be fixed and that the lever turns about C , the principle of virtual work proves that

in *rotation* the sum of the moments, about the axis of rotation, of all the forces acting on the rotating body, is zero. Again, if the forces acting on the lever above are *parallel*, it is readily proved, by the principle of virtual work, that the magnitudes of the forces P and Q are inversely as their distances from the fulcrum.

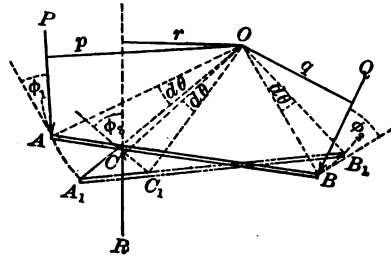


FIG. 74.

Sign is given to the work in accordance with whether the displacement is *in* or *opposite* to the direction of the component of the force, plus when *in* and minus when *opposite* to.

MOMENTS OF INERTIA.

Referring to Chapter XI, we find that the moment of the inertia force for any mass suffering an angular acceleration about any axis is $\Sigma r^2 dm \frac{d^2\theta}{dt^2}$. The factor $\Sigma r^2 dm$, denoted by I , is commonly called the moment of inertia of the mass and its value depends upon the shape of the mass. The moment of inertia of an element dm of mass, with reference to any line, is the product of the mass and the square of the distance of the element from the line; or, locating the position of the element of mass by its distances, x , y and z , from co-ordinate planes, the moments of inertia of any whole mass, with reference to the axes of X , Y and Z , are found by passing to the limit and summing up the moments of inertia for the separate particles; thus

$$I_x = \int (y^2 + z^2) dm, \quad I_y = \int (x^2 + z^2) dm, \quad I_z = \int (x^2 + y^2) dm. \quad (1)$$

The squared distances from the respective axes being in each case the sum of the squares of the *remaining* co-ordinates of the element *i. e.*, the squared distance, r^2 , from the X axis is $(y^2 + z^2)$ from the figure.

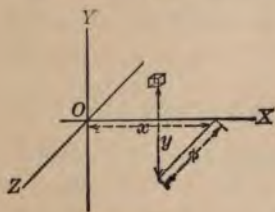


FIG. 75.

From (1) moments of inertia being obviously positive can never be equal to zero or a negative number.

In the above the *point element* has been used because each particle of the element is at a common (the same) distance from the axis of moments. Elements should be chosen with care so that this may *always* be the case.

If we suppose a line to be a single row of particles of matter, of which there are q per unit of length, the moment of inertia of a

straight line with reference to an axis perpendicular to it through its end is

$$I = \int_0^l x^2 q dx = \frac{ql^3}{3}.$$

Now the whole mass (number of particles) of the line is ql (q is always constant for any particular mass unless the density varies), therefore the mean squared distance of all the particles in the line from the axis of moments is $\frac{l^2}{3}$. The mean of the squared distances of all the particles in any mass from the axis of moments is called the *squared radius of gyration*, and is usually designated by k^2 . Thus the moment of inertia of any body whose whole mass, $\frac{W}{g}$, is M , is k^2M , where k^2 is the *squared radius of gyration*.

If we suppose a plane area to be a lamina of the uniform thickness of one particle of matter and that the axis of moments lies in the plane area, we may show that the squared radius of gyration for a rectangle, b wide and l long, for an axis of moments which is the side b , is $\frac{l^2}{3}$. For the element used (the same as that used in the c. g. method) is made up of particles which are all at the same distance from the axis of moments. The moment of inertia is $\frac{l^2}{3} \cdot qb \cdot ql = k^2M$. As a matter of fact, areas have no thickness and therefore M is considered to be the area itself, $bl = A$, and thus the moment of inertia of an area is k^2A . Therefore, in the above, the moment of inertia of the rectangle, about the side b is $\frac{l^2}{3} \cdot bl$, and, by analogy, the moment of inertia of the rectangle about the side l is $\frac{b^2}{3} \cdot bl$.

The moment of inertia of an area, about an axis perpendicular to the plane of the area and through the origin of co-ordinates for

points on the curve bounding the area, is called the *polar moment of inertia of the area*.

Referring to Fig. 76, the moment of inertia of the element of area, $dxdy$, about the Y axis, is x^2dxdy ; about the X axis it is y^2dxdy ; and the *polar moment of inertia of the element* is $(x^2 + y^2)dxdy$. The polar moment of inertia is thus the sum of the moments of inertia of the elementary area about the X and Y axes, and obviously the *polar squared radius of gyration* is

$$k_p^2 = k_x^2 + k_y^2. \quad (2)$$

Referring to equation (1) we see that for *any solid* the squared distance of an element $dm = dxdydz$, from any one co-ordinate axis, is equal to the sum of the squares of the distances of the element from the co-ordinate planes whose intersection is in that axis.

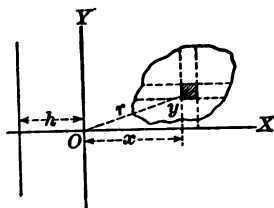


FIG. 76.

Again, referring to Fig. 76, the moment of inertia of the element of area, $dxdy$, about an axis parallel to the axis of Y and at a distance h from it, is $(x+h)^2dxdy$; or, $x^2dxdy + 2hxdxdy + h^2dxdy$. Integrating these elements over the whole area,

$$k_x^2A + 2h \int x dA + h^2A.$$

Now the factor $\int x dA$, of the middle term, is the statical moment of the area with reference to the Y axis, and if the c. g. of the area lies on the Y axis, this factor becomes zero and the middle term will vanish; therefore, having the moment of inertia, k_o^2A , of an area about an axis through its c. g., the moment of inertia of the area about an axis parallel to the "gravity axis" and at a distance h from it, is equal to $(k_o^2 + h^2)A$; or, the squared radius

of gyration, k^2 , of any area about an axis *parallel* to a gravity axis of the area and at a distance h from the gravity axis, is

$$k^2 = k_0^2 + h^2. \quad (3)$$

From (3) it is obvious that the moment of inertia with respect to a gravity axis is less than the moment of inertia with respect to any axis parallel to it.

Referring to Fig. 77, we will find the moment of inertia of the area of the circle with respect to the diameter lying in the Y axis. The equation of the circle being

$$x^2 + y^2 = a^2.$$

Using the point element, we have

$$\begin{aligned} I_y &= \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} x^2 \cdot dx dy \\ &= 2 \int_{-a}^a x^2 \sqrt{a^2-x^2} \cdot dx. \end{aligned}$$

Substituting $x = a \cos \theta$, whence

$$dx = -a \sin \theta d\theta,$$

$$I_y = 4a^4 \int_0^{\frac{\pi}{2}} \cos^2 \theta \sin^2 \theta d\theta = \frac{\pi a^4}{4}.$$

From which, the area of the circle being πa^2 , the squared radius of gyration for a circle of radius a about a diameter, is

$$k^2 = \frac{a^2}{4}.$$

As this is true for any diameter, it is true for the one lying in the X axis, and, therefore, the polar squared radius of gyration is (by 2)

$$k_p^2 = \frac{a^2}{4} + \frac{a^2}{4} = \frac{a^2}{2}, \text{ and } I_p = \frac{\pi a^4}{2}.$$

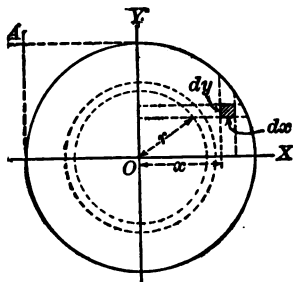


FIG. 77.

A simple way of finding the polar moment of inertia for the circle is by using polar co-ordinates and selecting a circumferential element of area equal to $2\pi r dr$, which has each of its points (particles) at the constant distance r from the polar axis. With this element (see figure)

$$I_p = \int_0^a r^2 2\pi r dr = 2\pi \left[\frac{r^4}{4} \right]_0^a = \frac{\pi a^4}{2}.$$

Again, from equation (3), noting that a diameter is a gravity axis, the moment of inertia of the area of a circle about a *tangent* to the circle, is

$$k_t^2 = \frac{a^2}{4} + a^2 = \frac{5a^2}{4}, \text{ whence } I_t = \frac{5\pi a^4}{4}.$$

Noting that k_t^2 for a tangent perpendicular to this tangent will be of equal value, the moment of inertia of the area of the circle about an axis through A and perpendicular to the plane of the circle, is by equation (2)

$$k_A^2 = \frac{5a^2}{4} + \frac{5a^2}{4} = \frac{5a^2}{2}, \text{ whence } I_A = \frac{5\pi a^4}{2}.$$

This latter value for k_A^2 comes directly from equation (3), also, if we find k^2 for an axis parallel to the polar axis and at a distance of $\sqrt{a^2 + a^2}$, from it in accordance with (3)

$$k_A^2 = \frac{a^2}{2} + 2a^2 = \frac{5a^2}{2}, \text{ whence } I_A = \frac{5\pi a^4}{2}, \text{ as above.}$$

These equations, showing the relation between the different radii of gyration, are useful when finding the moments of inertia of solids, as:

Referring to Fig. 78, it is required to find the moment of inertia

of the right cone shown, about an axis through its vertex and perpendicular to its geometric axis. Assume this axis to be the Y axis and the geometric axis the X axis, then, for a relation between x and y , we have

$$\frac{y}{x} = \frac{b}{h}, \text{ or } y = \frac{b}{h} x.$$

The squared radius of gyration of the element shown (a circular lamina) about its vertical diameter, is $\frac{y^2}{4}$, and about a parallel axis at a distance x from this diameter,

$$k^2 = \frac{y^2}{4} + x^2,$$

hence the moment of inertia of the element about the required axis is $\left(\frac{y^2}{4} + x^2\right) \pi y^2 dx$. Summing up these moments for all the elements (integrating), we have

$$\begin{aligned} I_y &= \pi \int_0^h \left(\frac{y^2}{4} + x^2 \right) y^2 dx = \pi \int_0^h \frac{b^2}{h^2} \left(\frac{b^2}{4h^2} + 1 \right) x^4 dx \\ &= \frac{\pi b^2}{4h^4} (b^2 + 4h^2) \frac{x^5}{5} \Big|_0^h = \frac{\pi b^2 h}{20} (b^2 + 4h^2). \end{aligned}$$

The volume of the cone is $\frac{\pi b^2 h}{3}$, and therefore the squared radius of gyration of the cone about the required axis, is

$$k^2 = \frac{I_y}{V} = \frac{3}{20} (b^2 + 4h^2).$$

This method is simpler than using the point element and the equation of the conical surface and can be employed frequently; that is, with all solids of revolution and also in cases like the following:

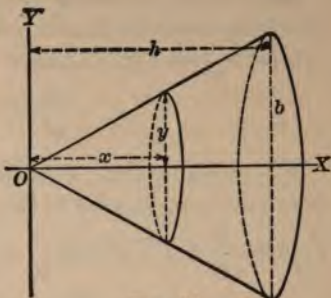


FIG. 78.

Let us find the moment of inertia and the squared radius of gyration, about the Y axis, for the ellipsoid, the equation of whose surface is (see Fig. 73 under centers of gravity)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

When finding the c. g. of half this solid, we found the semi-axes of any section, parallel to the YZ plane, and at a distance x from it, to be $\frac{b}{a} \sqrt{a^2 - x^2}$ and $\frac{c}{a} \sqrt{a^2 - x^2}$. The area of the elementary section is, therefore, $\frac{\pi bc}{a^2} (a^2 - x^2)$. The squared radius of gyration for an ellipse about one of its axes is the square of *the other semi-axis* divided by 4; therefore, the squared radius of gyration for the elementary volume (Fig. 73) about its vertical axis is $\frac{c^2}{4a^2} (a^2 - x^2)$ and for a parallel axis at a distance x from this vertical axis of the element, k^2 is $\frac{c^2}{4a^2} (a^2 - x^2) + x^2$. The moment of inertia of the element about the required axis is, therefore,

$$\left\{ \frac{c^2}{4a^2} (a^2 - x^2) + x^2 \right\} \frac{\pi bc}{a^2} (a^2 - x^2) dx,$$

and

$$\begin{aligned} I_y &= \frac{\pi bc}{a^2} \int_{-a}^a \left\{ \frac{c^2}{4a^2} (a^2 - x^2)^2 + x^2 (a^2 - x^2) \right\} dx \\ &= \frac{4}{15} \pi abc (c^2 + a^2). \end{aligned}$$

The volume of the ellipsoid is $\frac{4}{3} \pi abc$ and hence

$$k_y^2 = \frac{1}{5} (a^2 + c^2).$$

By analogy, $k_x^2 = \frac{1}{5} (b^2 + c^2)$ and $k_z^2 = \frac{1}{5} (a^2 + b^2)$.

If the semi-axes are all equal we have a sphere. Hence the k^2 for a sphere of radius a about a diameter is $\frac{2}{5} a^2$.

FORMULÆ.

TRIGONOMETRY.

$$\sin^2 \theta = 1 - \cos^2 \theta; \sec^2 \theta = 1 + \tan^2 \theta.$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y.$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

$$\sin 2x = 2 \sin x \cos x; \cos 2x = \cos^2 x - \sin^2 x; \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x); \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

$$2 \sin^2 \frac{x}{2} = 1 - \cos x; 2 \cos^2 \frac{x}{2} = 1 + \cos x.$$

$$\sin x + \sin y = 2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y).$$

$$\sin x - \sin y = 2 \cos \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y).$$

$$\cos x + \cos y = 2 \cos \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y).$$

$$\cos x - \cos y = -2 \sin \frac{1}{2}(x+y) \sin \frac{1}{2}(x-y).$$

$$\sin(x+y) + \sin(x-y) = 2 \sin x \cos y.$$

$$\sin(x+y) - \sin(x-y) = 2 \cos x \sin y.$$

$$\cos(x+y) + \cos(x-y) = 2 \cos x \cos y.$$

$$\cos(x+y) - \cos(x-y) = -2 \sin x \sin y.$$

$$f\left(n \frac{\pi}{2} \pm x\right) = \pm \text{co. } f(x) \text{ when } n \text{ is odd.}$$

$$f\left(n \frac{\pi}{2} \pm x\right) = \pm f(x) \text{ when } n \text{ is even.}$$

$$f(2n\pi + x) = f(x).$$

	0°	30°	45°	60°	15°	18°	22½°	36°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2}-\sqrt{2}}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2}+\sqrt{2}}{2}$	$\frac{\sqrt{5}+1}{4}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$2-\sqrt{3}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{2}-1$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	∞

From this table the functions of 75° , 72° , $67\frac{1}{2}^\circ$, 54° may be taken, these angles being $\frac{\pi}{2}$ minus 15° , 18° , $22\frac{1}{2}^\circ$ and 36° , respectively.

DIFFERENTIAL CALCULUS.

Differentials.

$$d(x+y) = dx + dy, \quad d(x-y) = dx - dy.$$

$$d(xy) = ydx + xdy.$$

$$d\left(\frac{1}{x}\right) = -\frac{dx}{x^2}.$$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}.$$

$$d(x^n) = nx^{n-1} dx.$$

$$d(\log_e x) = \frac{dx}{x}, \quad d(\log_a x) = \log_a e \cdot \frac{dx}{x}.$$

$$d(e^x) = e^x dx, \quad d(a^x) = a^x \cdot \log_e a \cdot dx.$$

$$d(y^x) = xy^{x-1} dy + \log y \cdot y^x \cdot dx.$$

$$d(\sin x) = \cos x dx.$$

$$d(\cos x) = -\sin x dx.$$

$$d(\tan x) = \sec^2 x dx.$$

$$d(\cot x) = -\operatorname{cosec}^2 x dx.$$

$$d(\sec x) = \sec x \tan x dx.$$

$$d(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x dx.$$

$$d(\operatorname{vers} x) = \sin x dx.$$

$$d(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}}.$$

$$d(\cos^{-1} x) = -\frac{dx}{\sqrt{1-x^2}}.$$

$$d(\tan^{-1} x) = \frac{dx}{1+x^2}.$$

$$d(\cot^{-1} x) = -\frac{dx}{1+x^2}.$$

$$d(\sec^{-1} x) = \frac{dx}{x\sqrt{x^2-1}}.$$

$$d(\operatorname{cosec}^{-1} x) = -\frac{dx}{x\sqrt{x^2-1}}.$$

$$d(\operatorname{vers}^{-1} x) = \frac{dx}{\sqrt{2x-x^2}}.$$

$$\text{For arc } ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta.$$

$$\text{Radius of curvature } \rho = \frac{ds}{d\phi} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}.$$

INTEGRAL CALCULUS.

General Integrals.

$$\int x^n dx = \frac{x^{n+1}}{n+1}.$$

$$\int \frac{dx}{x^n} = -\frac{1}{(n-1)x^{n-1}}.$$

$$\int \frac{dx}{x} = \log(\pm x).$$

$$\int a^x dx = \frac{a^x}{\log a}. \quad \int e^x dx = e^x. \quad \int e^{ax} dx = \frac{e^{ax}}{a}.$$

$$\int \cos \theta d\theta = \sin \theta.$$

$$\int \cos^2 \theta \, d\theta = \frac{1}{2}(\theta + \sin \theta \cos \theta).$$

$$\int \sin \theta \, d\theta = -\cos \theta.$$

$$\int \sin^2 \theta \, d\theta = \frac{1}{2}(\theta - \sin \theta \cos \theta).$$

$$\int \sec^2 \theta \, d\theta = \tan \theta.$$

$$\int \operatorname{cosec}^2 \theta \, d\theta = -\cot \theta.$$

$$\int \sec \theta \tan \theta \, d\theta = \sec \theta.$$

$$\int \operatorname{cosec} \theta \cot \theta \, d\theta = -\operatorname{cosec} \theta.$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = -\cos^{-1} x.$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} = -\cos^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x = -\cot^{-1} x.$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} = -\frac{1}{a} \cot^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x = -\operatorname{cosec}^{-1} x.$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} = -\frac{1}{a} \operatorname{cosec}^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{\sqrt{2x-x^2}} = \operatorname{vers}^{-1} x.$$

$$\int \frac{dx}{\sqrt{2ax-x^2}} = \operatorname{vers}^{-1} \frac{x}{a}.$$

$$\int \sin^3 \theta \, d\theta = \int (1 - \cos^2 \theta) \sin \theta \, d\theta = -\cos \theta + \frac{1}{3} \cos^3 \theta.$$

$$\int \cos^3 \theta \, d\theta = \int (1 - \sin^2 \theta) \cos \theta \, d\theta = \sin \theta - \frac{1}{3} \sin^3 \theta.$$

$$d(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}}.$$

$$d(\cos^{-1} x) = -\frac{dx}{\sqrt{1-x^2}}.$$

$$d(\tan^{-1} x) = \frac{dx}{1+x^2}.$$

$$d(\cot^{-1} x) = -\frac{dx}{1+x^2}.$$

$$d(\sec^{-1} x) = \frac{dx}{x\sqrt{x^2-1}}.$$

$$d(\operatorname{cosec}^{-1} x) = -\frac{dx}{x\sqrt{x^2-1}}.$$

$$d(\operatorname{vers}^{-1} x) = \frac{dx}{\sqrt{2x-x^2}}.$$

$$\text{For arc } ds = \sqrt{\overline{dx^2} + \overline{dy^2}} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \cdot d\theta.$$

$$\text{Radius of curvature } \rho = \frac{ds}{d\phi} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2r}{d\theta^2}}.$$

INTEGRAL CALCULUS.

General Integrals.

$$\int x^n dx = \frac{x^{n+1}}{n+1}.$$

$$\int \frac{dx}{x^n} = -\frac{1}{(n-1)x^{n-1}}.$$

$$\int \frac{dx}{x} = \log(\pm x).$$

$$\int a^x dx = \frac{a^x}{\log a}. \quad \int e^x dx = e^x. \quad \int e^{ax} dx = \frac{e^{ax}}{a}.$$

$$\int \cos \theta d\theta = \sin \theta.$$

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta = \frac{(n-1)(n-3)\dots 1}{n(n-2)\dots 2} \cdot \frac{\pi}{2} \text{ if } n \text{ is even.}$$

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \, d\theta = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta = \frac{(n-1)(n-3)\dots 2}{n(n-2)\dots 3} \text{ if } n \text{ is odd.}$$

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \cos^m \theta \, d\theta = \frac{(m-1)(m-3)\dots (n-1)(n-3)\dots}{(m+n)(m+n-2)\dots}$$

if either n or m is odd.

$$\int_0^{\frac{\pi}{2}} \sin^n \theta \cos^m \theta \, d\theta = \frac{(m-1)(m-3)\dots 1 \cdot (n-1)(n-3)\dots 1}{(m+n)(m+n-2)\dots 2} \cdot \frac{\pi}{2}$$

if both n and m are even.

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.$$

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

$$\int \frac{dx}{\sin^n x} = -\frac{1}{n-1} \cdot \frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}.$$

$$\int \frac{dx}{\cos^n x} = \frac{1}{n-1} \cdot \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} x}.$$

$$\int \cos^m x \sin^n x \, dx = \frac{\cos^{m-1} x \sin^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \cos^{m-2} x \sin^n x \, dx.$$

$$\int \cos^m x \sin^n x \, dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n} \int \cos^m x \sin^{n-2} x \, dx.$$

$$\int \frac{dx}{\sin^m x \cos^n x} = \frac{1}{n-1} \cdot \frac{1}{\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m x \cos^{n-2} x}.$$

$$\int \frac{dx}{\sin^m x \cos^n x} = -\frac{1}{m-1} \cdot \frac{1}{\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{m-1} \int \frac{dx}{\sin^{m-2} x \cos^n x}.$$

$$\int \frac{\cos^m x dx}{\sin^n x} = -\frac{\cos^{m+1} x}{(n-1)\sin^{n-1} x} - \frac{m-n+2}{n-1} \int \frac{\cos^m x dx}{\sin^{n-2} x},$$

$$\int \frac{\cos^m x dx}{\sin^n x} = \frac{\cos^{m-1} x}{(m-n)\sin^{n-1} x} + \frac{m-1}{m-n} \int \frac{\cos^{m-2} x dx}{\sin^n x}.$$

Miscellaneous Integrals.

$$\int \frac{dx}{a+bx} = \frac{1}{b} \log(a+bx).$$

$$\int (a+bx)^n dx = \frac{(a+bx)^{n+1}}{b(n+1)}.$$

$$\int \frac{x dx}{a+bx} = \frac{1}{b^2} \{a+bx - a \log(a+bx)\}.$$

$$\int \frac{x^2 dx}{a+bx} = \frac{1}{b^3} \left\{ \frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2 \log(a+bx) \right\}.$$

$$\int \frac{x dx}{(a+bx)^2} = \frac{1}{b^2} \left\{ \log(a+bx) + \frac{a}{a+bx} \right\}.$$

$$\int \frac{x^2 dx}{(a+bx)^2} = \frac{1}{b^3} \left\{ a+bx - 2a \log(a+bx) - \frac{a^2}{a+bx} \right\}.$$

$$\int \frac{x dx}{(a+bx)^3} = \frac{1}{b^2} \left\{ \frac{a}{2(a+bx)^2} - \frac{1}{a+bx} \right\}.$$

$$\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2} \log \left(\frac{a+bx}{x} \right).$$

$$\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2} \log \left(\frac{a+bx}{x} \right).$$

$$\int \frac{dx}{x(a+bx)} = -\frac{1}{a} \log \left(\frac{a+bx}{x} \right).$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1} x \sqrt{\frac{b}{a}}; a \text{ and } b \text{ positive}.$$

$$\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab} \log \left(\frac{a+bx}{a-bx} \right).$$

$$\int \frac{dx}{x(a+bx^n)} = \frac{1}{an} \log \left(\frac{x^n}{a+bx^n} \right).$$

$$\int \frac{x dx}{a+bx^2} = \frac{1}{2b} \log \left(x^2 + \frac{a}{b} \right).$$

$$\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a} \log \left(\frac{x^2}{a+bx^2} \right).$$

$$\int \frac{x^2 dx}{a+bx^2} = \frac{x}{b} - \frac{a^{\frac{1}{2}}}{b^{\frac{3}{2}}} \tan^{-1} x \sqrt{\frac{b}{a}}; a \text{ and } b \text{ positive.}$$

$$\int x(a+bx)^{\frac{1}{2}} dx = -\frac{(4a-6bx)\sqrt{(a+bx)^3}}{15b^2}.$$

$$\int x^2(a+bx)^{\frac{1}{2}} dx = \frac{(16a^2-24abx+30b^2x^2)\sqrt{(a+bx)^3}}{105b^3}.$$

$$\int \frac{xdx}{(a+bx)^{\frac{3}{2}}} = -\frac{(4a-2bx)\sqrt{a+bx}}{3b^2}.$$

$$\int \frac{x^2 dx}{(a+bx)^{\frac{3}{2}}} = \frac{(16a^2-8abx+6b^2x^2)\sqrt{a+bx}}{15b^3}.$$

$$\int \frac{dx}{x(a+bx)^{\frac{1}{2}}} = \frac{1}{\sqrt{a}} \log \frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}; a \text{ positive.}$$

$$\int \frac{dx}{x(a+bx)^{\frac{1}{2}}} = \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bx}{-a}}; a \text{ negative.}$$

$$\int (x^2+a^2)^{\frac{1}{2}} dx = \frac{x}{8} (2x^2+5a^2) \sqrt{x^2+a^2} + \frac{3a^4}{8} \log(x+\sqrt{x^2+a^2}).$$

$$\int x(x^2+a^2)^{\frac{n}{2}} dx = \frac{(x^2+a^2)^{\frac{n+2}{2}}}{n+2}.$$

$$\int x^2(x^2+a^2)^{\frac{1}{2}} dx = \frac{x}{8} (2x^2+a^2) \sqrt{x^2+a^2} - \frac{a^4}{8} \log(x+\sqrt{x^2+a^2}).$$

$$\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{x}{a^2 \sqrt{x^2+a^2}}.$$

$$\int \frac{xdx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}.$$

$$\int \frac{x^2 dx}{\sqrt{x^2+a^2}} = \frac{x}{2} \sqrt{x^2+a^2} - \frac{a^2}{2} \log(x+\sqrt{x^2+a^2}).$$

$$\int \frac{x^2 dx}{(x^2+a^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2+a^2}} + \log(x+\sqrt{x^2+a^2}).$$

$$\int \frac{dx}{x(x^2+a^2)^{\frac{1}{2}}} = \frac{1}{a} \log \left(\frac{x}{a+\sqrt{x^2+a^2}} \right).$$

$$\int \frac{dx}{x^2(x^2+a^2)^{\frac{1}{2}}} = -\frac{\sqrt{x^2+a^2}}{a^2x}.$$

$$\int \frac{dx}{x^3(x^2+a^2)^{\frac{1}{2}}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3} \log \left(\frac{a+\sqrt{x^2+a^2}}{x} \right).$$

$$\int \frac{x dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}.$$

$$\int \frac{(x^2+a^2)^{\frac{1}{2}} dx}{x} = \sqrt{x^2+a^2} - a \log \left(\frac{a+\sqrt{x^2+a^2}}{x} \right).$$

$$\int \frac{(x^2+a^2)^{\frac{1}{2}} dx}{x^2} = -\frac{\sqrt{x^2+a^2}}{x} + \log(x+\sqrt{x^2+a^2}).$$

$$\int (x^2-a^2)^{\frac{1}{2}} dx = \frac{x}{8} (2x^2-5a^2) \sqrt{x^2-a^2} + \frac{3a^4}{8} \log(x+\sqrt{x^2-a^2}).$$

$$\int x(x^2-a^2)^{\frac{n}{2}} dx = \frac{(x^2-a^2)^{\frac{n+2}{2}}}{n+2}.$$

$$\int x^2(x^2-a^2)^{\frac{1}{2}} dx = \frac{x}{8} (2x^2-a^2) \sqrt{x^2-a^2} - \frac{a^4}{8} \log(x+\sqrt{x^2-a^2}).$$

$$\int \frac{x^2 dx}{\sqrt{x^2-a^2}} = \frac{x}{2} \sqrt{x^2-a^2} + \frac{a^2}{2} \log(x+\sqrt{x^2-a^2}).$$

$$\int \frac{x^2 dx}{(x^2-a^2)^{\frac{1}{2}}} = -\frac{x}{\sqrt{x^2-a^2}} + \log(x+\sqrt{x^2-a^2}).$$

$$\int \frac{dx}{x^2(x^2-a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2-a^2}}{a^2x}.$$

$$\int \frac{dx}{x^3(x^2-a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3} \sec^{-1} \frac{x}{a}.$$

$$\int \frac{(x^2-a^2)^{\frac{1}{2}} dx}{x} = \sqrt{x^2-a^2} - a \cos^{-1} \frac{a}{x}.$$

$$\int \frac{(x^2-a^2)^{\frac{1}{2}} dx}{x^2} = -\frac{\sqrt{x^2-a^2}}{x} + \log(x+\sqrt{x^2-a^2}).$$

$$\int (a^2-x^2)^{\frac{1}{2}} dx = \frac{x}{8} (5a^2-2x^2) \sqrt{a^2-x^2} + \frac{3a^4}{8} \sin^{-1} \frac{x}{a}.$$

$$\int x(a^2-x^2)^{\frac{n}{2}} dx = -\frac{(a^2-x^2)^{\frac{n+2}{2}}}{n+2}.$$

$$\int \frac{dx}{x(a+bx^2)} = \frac{1}{2a} \log \left(\frac{x^2}{a+bx^2} \right).$$

$$\int \frac{x^2 dx}{a+bx^2} = \frac{x}{b} - \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} \tan^{-1} x \sqrt{\frac{b}{a}}; a \text{ and } b \text{ positive.}$$

$$\int x(a+bx)^{\frac{1}{2}} dx = -\frac{(4a-6bx)\sqrt{(a+bx)^3}}{15b^2}.$$

$$\int x^2(a+bx)^{\frac{1}{2}} dx = \frac{(16a^2-24abx+30b^2x^2)\sqrt{(a+bx)^3}}{105b^3}.$$

$$\int \frac{xdx}{(a+bx)^{\frac{3}{2}}} = -\frac{(4a-2bx)\sqrt{a+bx}}{3b^2}.$$

$$\int \frac{x^2 dx}{(a+bx)^{\frac{3}{2}}} = \frac{(16a^2-8abx+6b^2x^2)\sqrt{a+bx}}{15b^3}.$$

$$\int \frac{dx}{x(a+bx)^{\frac{1}{2}}} = \frac{1}{\sqrt{a}} \log \frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}; a \text{ positive.}$$

$$\int \frac{dx}{x(a+bx)^{\frac{1}{2}}} = \frac{2}{\sqrt{-a}} \tan^{-1} \sqrt{\frac{a+bx}{-a}}; a \text{ negative.}$$

$$\int (x^2+a^2)^{\frac{1}{2}} dx = \frac{x}{8} (2x^2+5a^2) \sqrt{x^2+a^2} + \frac{3a^4}{8} \log(x+\sqrt{x^2+a^2}).$$

$$\int x(x^2+a^2)^{\frac{n}{2}} dx = \frac{(x^2+a^2)^{\frac{n+2}{2}}}{n+2}.$$

$$\int x^2(x^2+a^2)^{\frac{1}{2}} dx = \frac{x}{8} (2x^2+a^2) \sqrt{x^2+a^2} - \frac{a^4}{8} \log(x+\sqrt{x^2+a^2}).$$

$$\int \frac{dx}{(x^2+a^2)^{\frac{3}{2}}} = \frac{x}{a^2 \sqrt{x^2+a^2}}.$$

$$\int \frac{xdx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}.$$

$$\int \frac{x^2 dx}{\sqrt{x^2+a^2}} = \frac{x}{2} \sqrt{x^2+a^2} - \frac{a^2}{2} \log(x+\sqrt{x^2+a^2}).$$

$$\int \frac{x^2 dx}{(x^2+a^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2+a^2}} + \log(x+\sqrt{x^2+a^2}).$$

$$\int \frac{dx}{x(x^2+a^2)^{\frac{1}{2}}} = \frac{1}{a} \log \left(\frac{x}{a+\sqrt{x^2+a^2}} \right).$$

$$\int \frac{dx}{x^2(x^2+a^2)^{\frac{1}{2}}} = -\frac{\sqrt{x^2+a^2}}{a^2x}.$$

$$\int \frac{dx}{x^3(x^2+a^2)^{\frac{1}{2}}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3} \log \left(\frac{a+\sqrt{x^2+a^2}}{x} \right).$$

$$\int \frac{x dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}.$$

$$\int \frac{(x^2+a^2)^{\frac{1}{2}} dx}{x} = \sqrt{x^2+a^2} - a \log \left(\frac{a+\sqrt{x^2+a^2}}{x} \right).$$

$$\int \frac{(x^2+a^2)^{\frac{1}{2}} dx}{x^2} = -\frac{\sqrt{x^2+a^2}}{x} + \log(x+\sqrt{x^2+a^2}).$$

$$\int (x^2-a^2)^{\frac{1}{2}} dx = \frac{x}{8} (2x^2-5a^2) \sqrt{x^2-a^2} + \frac{3a^4}{8} \log(x+\sqrt{x^2+a^2}).$$

$$\int x(x^2-a^2)^{\frac{n}{2}} dx = \frac{(x^2-a^2)^{\frac{n+2}{2}}}{n+2}.$$

$$\int x^2(x^2-a^2)^{\frac{1}{2}} dx = \frac{x}{8} (2x^2-a^2) \sqrt{x^2-a^2} - \frac{a^4}{8} \log(x+\sqrt{x^2-a^2}).$$

$$\int \frac{x^2 dx}{\sqrt{x^2-a^2}} = \frac{x}{2} \sqrt{x^2-a^2} + \frac{a^2}{2} \log(x+\sqrt{x^2-a^2}).$$

$$\int \frac{x^2 dx}{(x^2-a^2)^{\frac{1}{2}}} = -\frac{x}{\sqrt{x^2-a^2}} + \log(x+\sqrt{x^2-a^2}).$$

$$\int \frac{dx}{x^2(x^2-a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2-a^2}}{a^2x}.$$

$$\int \frac{dx}{x^3(x^2-a^2)^{\frac{1}{2}}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3} \sec^{-1} \frac{x}{a}.$$

$$\int \frac{(x^2-a^2)^{\frac{1}{2}} dx}{x} = \sqrt{x^2-a^2} - a \cos^{-1} \frac{a}{x}.$$

$$\int \frac{(x^2-a^2)^{\frac{1}{2}} dx}{x^2} = -\frac{\sqrt{x^2-a^2}}{x} + \log(x+\sqrt{x^2-a^2}).$$

$$\int (a^2-x^2)^{\frac{1}{2}} dx = \frac{x}{8} (5a^2-2x^2) \sqrt{a^2-x^2} + \frac{3a^4}{8} \sin^{-1} \frac{x}{a}.$$

$$\int x(a^2-x^2)^{\frac{n}{2}} dx = -\frac{(a^2-x^2)^{\frac{n+2}{2}}}{n+2}.$$

$$\int x^2 (a^2 - x^2)^{\frac{1}{2}} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \sin^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \frac{x}{a^2 \sqrt{a^2 - x^2}}.$$

$$\int \frac{x^2 dx}{(a^2 - x^2)^{\frac{3}{2}}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}.$$

$$\int \frac{x^2 dx}{(a^2 - x^2)^{\frac{5}{2}}} = \frac{x}{\sqrt{a^2 - x^2}} - \sin^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{x(a^2 - x^2)^{\frac{3}{2}}} = \frac{1}{a} \log \left(\frac{x}{a + \sqrt{a^2 - x^2}} \right).$$

$$\int \frac{dx}{x^2 (a^2 - x^2)^{\frac{3}{2}}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x}.$$

$$\int \frac{dx}{x^3 (a^2 - x^2)^{\frac{3}{2}}} = -\frac{\sqrt{a^2 - x^2}}{2a^2 x^2} + \frac{1}{2a^3} \log \left(\frac{x}{a + \sqrt{a^2 - x^2}} \right).$$

$$\int \frac{(a^2 - x^2)^{\frac{1}{2}} dx}{x} = \sqrt{a^2 - x^2} - a \log \left(\frac{a + \sqrt{a^2 - x^2}}{x} \right).$$

$$\int \frac{(a^2 - x^2)^{\frac{1}{2}} dx}{x^2} = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1} \frac{x}{a}.$$

$$\int \sqrt{2ax - x^2} \cdot dx = \frac{x-a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \text{vers}^{-1} \frac{x}{a}.$$

$$\int \sqrt{2ax - x^2} \cdot x dx = -\frac{3a^2 + ax - 2x^2}{6} \sqrt{2ax - x^2} + \frac{a^3}{2} \text{vers}^{-1} \frac{x}{a}.$$

$$\int \frac{dx}{x \sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{ax}.$$

$$\int \frac{x dx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a \text{vers}^{-1} \frac{x}{a}.$$

$$\int \frac{x^2 dx}{\sqrt{2ax - x^2}} = -\frac{x+3a}{2} \sqrt{2ax - x^2} + \frac{3a^2}{2} \text{vers}^{-1} \frac{x}{a}.$$

$$\int \frac{\sqrt{2ax - x^2} dx}{x} = \sqrt{2ax - x^2} + a \text{vers}^{-1} \frac{x}{a}.$$

$$\int \frac{\sqrt{2ax - x^2} \cdot dx}{x^2} = -\frac{2}{x} \cdot \sqrt{2ax - x^2} - \text{vers}^{-1} \frac{x}{a}.$$

$$\int \frac{\sqrt{2ax-x^2} \cdot dx}{x^3} = -\frac{(2ax-x^2)^{\frac{1}{2}}}{3ax^2}.$$

$$\int \frac{dx}{(2ax-x^2)^{\frac{1}{2}}} = \frac{x-a}{a^2\sqrt{2ax-x^2}}.$$

$$\int \frac{xdx}{(2ax-x^2)^{\frac{1}{2}}} = \frac{x}{a\sqrt{2ax-x^2}}.$$

$$\int \frac{dx}{\sqrt{2ax+x^2}} = \log(x+a+\sqrt{2ax+x^2}).$$

$$\int \sin mx \sin nx dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)}.$$

$$\int \cos mx \cdot \cos nx \cdot dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)}.$$

$$\int \sin mx \cdot \cos nx \cdot dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)}.$$

$$\int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{1}{ab} \tan^{-1} \left(\frac{b \tan x}{a} \right).$$

$$\int e^x \sin x dx = \frac{e^x(\sin x - \cos x)}{2}. \quad \int e^x \cos x dx = \frac{e^x(\sin x + \cos x)}{2}.$$

$$\int e^{ax} \sin nx dx = \frac{e^{ax}(a \sin nx - n \cos nx)}{a^2 + n^2}.$$

$$\int e^{ax} \cos nx dx = \frac{e^{ax}(n \sin nx + a \cos nx)}{a^2 + n^2}.$$

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1).$$

$$\int \log x dx = x \log x - x.$$

$$\int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{1}{2!} \log^2 x + \frac{1}{3!} \log^3 x \dots$$

$$\int \frac{dx}{x \log x} = \log(\log x).$$

$$\int x^n \log x dx = x^{n+1} \left(\frac{\log x}{n+1} - \frac{1}{(n+1)^2} \right).$$

ANSWERS TO PROBLEMS.

Problems I.

1. $R = \sqrt{1025}$; $\theta = \tan^{-1} 0.8$.
2. $R = \sqrt{1525}$; $\theta = \tan^{-1} \frac{2\sqrt{3}}{7}$.
3. $R = \sqrt{3}$; $\theta = \tan^{-1} \frac{1}{\sqrt{3}}$.
4. $R = \sqrt{109 + 30\sqrt{2}}$; N. $\tan^{-1} \frac{3}{10\sqrt{2} + 3}$ E.
5. $D = 5\sqrt{13 - 6\sqrt{2}}$; N. $\tan^{-1} - (6\sqrt{2} + 9)$ W.
6. 60 ft.; 80 ft.; $\tan^{-1} \frac{3}{4}$ with AB .
7. $71\frac{1}{2}$ and $42\frac{1}{2}$.
8. N. E. courses were $\frac{5\sqrt{2}}{3}$ and $\frac{10\sqrt{2}}{3}$; N. W. courses, $\frac{10\sqrt{2}}{3}$
and $\frac{5\sqrt{2}}{3}$.
9. $R = 5.87$; N. $\tan^{-1} \frac{2.65}{5.1}$ E.
10. The 5th displacement was 583.7 ft., at angle with 16 ft. dis.,
 $\tan^{-1} - \frac{48.2}{137}$.

Problems II.

1. 9 lbs.; $\tan^{-1} \frac{-7.8}{-4.5}$.
2. $2\sqrt{78}$ lbs.; $\tan^{-1} \frac{20 + 26\sqrt{2}}{23}$.
3. 3 lbs.; 90° .
4. 202.21 lbs.; $\cos \alpha = \frac{-7.5}{202+}$; $\cos \beta = \frac{-200}{202+}$; $\cos \gamma = \frac{-27.5}{202+}$.
5. $\sqrt{37}$; $\tan^{-1} \frac{3\sqrt{3}}{11}$.

6. $\sqrt{61}$; $\tan^{-1} \frac{5\sqrt{3}}{13}$.
7. 0 lbs.-ft.
8. -1.32 lbs.-ft.
9. $R = 60\sqrt{18 - \sqrt{3}}$; $\cos \alpha = \frac{\sqrt{3} - 1}{2\sqrt{18 - \sqrt{3}}}$; $\cos \beta = \frac{-2}{15\sqrt{18 - \sqrt{3}}}$;
 $\cos \gamma = \frac{\sqrt{3} - 1}{2\sqrt{18 - \sqrt{3}}}$.
10. 75 lbs.

Problems III.

1. (1) 13 lbs., 3 $\frac{9}{13}$ in. from 9 lb. force; (2) 5 lbs., 9 $\frac{3}{5}$ in.
2. Zero.
3. 18 lbs.; $\bar{x} = 3$; $\bar{y} = 1$.
4. 20 lbs., 10 lbs., 10 lbs.
5. $\bar{x} = 4\frac{2}{3}$; $\bar{y} = 6\frac{1}{3}$.
6. At a point dividing the line joining the middle points of the parallel sides in the ratio of 7:15
7. Couple. Moment is $\sqrt{505}$.
9. 444 sq. in.
10. 25 lbs., 3.4 ft. from the point; 5 lbs., 4.2 ft. from the point.

Problems IV.

1. $200/\sqrt{3}$ lbs; $100/\sqrt{3}$ lbs.
2. 200 lbs.; $200\sqrt{3}$ lbs.; 30° .
3. $10\sqrt{3}$ tons; $5.4 +$ tons.
4. 140 lbs.
5. $\tan^{-1} 3/2$; $150/\sqrt{13}$ lbs.; $250/\sqrt{13}$ lbs.
6. 17 tons; $\sqrt{89}$ tons.
7. $\sin^{-1} W/R$.
8. $\cos^{-1} 4/5$ or $-24/25$.
9. 4 ft.
10. 90 lbs.

Problems V.

1. (1) 100 lbs.; (2) $117\frac{1}{2}$ lbs.; (3) 98 lbs.
2. $\tan^{-1} 15/8$.
3. $\frac{1}{2\sqrt{3}}$.
4. $\frac{1}{2\sqrt{3}}$.
5. $100\sqrt{3}$ lbs.; 60° with horizontal.
6. $50(1 - 1/\sqrt{3})$ lbs.; $50 \left(\frac{3 + \sqrt{3}}{\sqrt{10}} \right)$ lbs.; $\tan^{-1} \frac{1}{3}$.
7. $\frac{3}{4}$ its length.
8. 11,133 lbs.
9. $3\frac{1}{2}$.
10. $80 +$ lbs.-ft. moment.

Problems VI.

1. 5.4 tons.
2. 12.8 tons; 19.5 tons.
3. $3Wa/d$.
4. 1st segment, either 32° or 52° ; tension, either $\sqrt{356}$ lbs. or $\sqrt{697}$ lbs.
5. 1050 lbs.; 1400 lbs.; 840 lbs.
6. $60(-)$ tons; $57(+)$ tons.
7. $244(-)$ tons; 255 tons; 150 tons.
8. 120 ft.

Problems VII.

1. $22/15$ f/s/s.
2. $\sqrt{3}$ f/s.
3. N. 60° W.; $20\sqrt{3}$ f/s.
4. 10.65 f/s.
5. 82 f/s; 18 f/s.
6. $5\frac{1}{2}$ sec.; 121 ft.

7. 40 f/s.
8. $89\frac{1}{2}$ ft.
9. 9 f/s; $\tan^{-1} 4/3$ (3d Quad.).
10. 16.5 ft.

Problems VIII.

2. $5g$; $\tan^{-1} \frac{4}{3}$.
3. $\frac{2V}{g} \tan \alpha$. $2 \frac{VV}{g} \sin \theta$
4. Roughly, 283 f/s; 13.5 sec.
8. 1522 f/s; 0.111 f/s².
10. 196 ft.; 3.5 sec.

Problems IX.

1. g lbs.
2. 1/444898 (nearly).
3. $78\frac{1}{2}$ lbs.
4. $2(-)$; 64.
5. $k/24$ (k =density unit dist. from end).
6. 80 ft.; g f/s.
7. $\frac{1}{2}$ sec.; $\frac{1}{2}$ f/s.
8. 100 lbs.; $2g$ f/s.
9. 4 min.
10. 400 lbs.
11. Impulse 2.9; force 29 lbs.

Problems X.

1. 1237500 ft.-lbs.
2. 500000 ft.-lbs.
3. 46(+) HP.
4. 160 HP.
5. $3441.\ddot{7}\ddot{7}$ ft.
6. 144(-) f/s.
7. 7884800 ft.-lbs.

8. $\frac{5a}{18}$ from starting point toward last side cushion struck.
9. .0000000026 ft.
10. 224 HP.

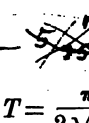
Problems XI.

1. $\frac{1}{4}$ lb. both ways.
2. 18*m*. lbs.
3. *m* lbs.
4. $m \frac{4a^2 V^2}{y^3}$.
5. $\frac{W\pi}{160}$ lbs.
6. $W/64$ lbs.
7. $m \frac{25\pi}{81}$ lbs.; $m \left(\frac{25\pi}{9}\right)^2 800$.
8. $\frac{W \times \pi^2}{32 \times 144}$; $\frac{W 125\pi^2}{72}$; 8 minutes.

Problems XII.

1. 209 lbs.; 176 lbs.; 132 lbs.
2. $\begin{cases} T \text{ max., } 5.27 \text{ tons; } T \text{ min., } 3.05 \text{ tons;} \\ \text{HP av., } 814.54; \text{HP max., } 944.38. \end{cases}$
3. $a = g/23$ f/s²; v , 2.8 f/s; ret., $g/21$ f/s²; t , 1.8 s.; space, 5.324 ft.
4. $a = g/13$ f/s²; T , 36/13 lbs.; space 240/13*g* ft.
5. $a = 3g/13$ f/s²; T , 30/13 lbs.
6. 3 73/125 HP.
7. 106.13 f/s.
8. 47920 ft.-lbs.
9. $20/\sqrt{15}$ lbs.
10. 3.55 tons.

Problems XIII.

1. Space $a \cos \left(\frac{Vt}{a} \right)$; time, $\frac{2\pi a}{V}$.
2. 128 f/s; 32 f/s.²
3. V max., $-\frac{g}{\sqrt{3}}$ 
4. Time, $\frac{\pi}{4g}$
5. ~~5/98 ft~~ 
6. v max. = $\sqrt{\frac{g}{3}}$; $T = \frac{\pi}{2\sqrt{3}g}$.
7. Time, $\frac{\pi}{2} \sqrt{\frac{R}{g}}$.
8. $v^2 = \frac{2gRh}{R+h}$.
9. $v^2 = V_0^2 e^{-2fs} - \frac{g}{f} (1 - e^{-2fs})$.
10. $s = \frac{1}{2f} \log \left(1 + \frac{f}{g} v_0^2 \right)$.

$\frac{\pi}{g\sqrt{3}}$

Problems XIV.

1. 2.127 g .
2. $\frac{4\pi^2}{5}$.
3. $\frac{g}{90}$.
4. Reduces g by .028.
5. 3.39 in.
6. $\tan^{-1} 1.07$.
7. $2\pi \sqrt{\frac{2}{g}}$ sec.
8. About .2 in.
9. $\sqrt{3g}$; $\sqrt{15g}$; $\sqrt{9g}$.
10. $\tan^{-1} .4$.

Problems XV.

1. 32.186 f/s^2 .
2. 31.28.
3. $\sqrt{gr}$.
4. $\frac{1}{1.007}$.
5. 0.6 in.
6. $4\pi \sqrt{\frac{a}{g}}$.
7. Lowered $\frac{l}{240}$.
8. Nearly $1\frac{5}{8}$ min.
9. $1\frac{1}{8}$.

Problems XVII.

1. $M = \frac{32\mu r \cdot W}{9\pi}$.
3. $\frac{112\pi}{297}$.
4. $\frac{56\pi}{297}$.
5. About 3000 lbs.
6. $179\frac{1}{2}$ HP.
7. .3 HP.
8. 65 lbs.
9. $T_1 = 86.5$; $T_2 = 39.3$ lbs.
10. 70 in.

Problems XVIII.

1. Nearly 2.9 sec.
2. 63.2 in.
3. (1) 22 f/s; (2) 31 f/s.
4. $3/2$ of a .
5. 53.7.

6. $\frac{mgb}{2}$.
7. 21 in.
8. 178.
9. Equal.
10. 2 ft.

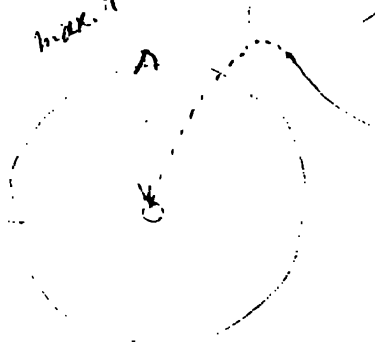
Problems XIX.

1. $\beta = \frac{128}{7}$; $T = \frac{100}{7}$; K. E. = 1632 ft.-lbs.
2. 44.2 turns.
3. 17.1 lbs.
4. 6.28 lbs.; K. E. $36\pi^2$.
5. $\beta = \frac{64}{289}$ radians; $\bar{\omega} = \frac{192}{289}$ radians.
6. $t = 10$ min. 42.2 sec.; 321.7 rev.
7. $\frac{3\pi^2}{20}$.
8. 105.6 lbs.
9. 13,200 lbs.-ft. units.
10. $10\frac{2}{11}$ rev.

Problems XX.

1. $s = \frac{50g}{3}$; $v = \frac{10g}{3}$; K. E. = $\frac{500g}{3}$.
2. $v = 2\sqrt{2g}$.
3. $a = \frac{5g}{14}$; $s = \frac{125g}{7}$.
4. $\frac{6250g}{49}$; $\frac{2500g}{49}$.
5. $\frac{3}{2}g \cdot \sin \alpha$.
6. $\frac{g}{2} \sin \alpha$.
7. $\frac{3}{2}g$.
8. 117.5 ft.-tons.

max. $\gamma = .85R$



at P cont.
but der $\frac{dP}{dr}$ is discount
at $\gamma = R$

ca. hyperbola

